Review Problem for Midterm #1

Midterm I: 10- 10:50 a.m Friday (Sep. 20), Topics: 5.8 and 6.1-6.3
Office Hours before the midterm I: W 2-4 pm (Sep 18) , Th 3-5 pm (Sep 19) at UH 2080B
Solution to quiz can be found at `http://math.utoledo.edu/~mtsui/math1760/hw.html`
No calculator is allowed in the exam. You should know how to solve these problems without a calculator.

1. Evaluate the following indefinite integrals:

(a)

$$\int \frac{-2x^3 - x + 1}{x^2} dx$$

Solution. We can first simplify
$\frac{-2x^3-x+1}{x^2} = -2\frac{x^3}{x^2} - \frac{x}{x^2} + \frac{1}{x^2} = -2x - x^{-1} + x^{-2}$.
So $\int \frac{-2x^3-x+1}{x^2}dx = \int(-2x - x^{-1} + x^{-2})dx$
$= -2\int xdx - \int x^{-1}dx + \int x^{-2}dx = -2\cdot\frac{x^2}{2} - \ln|x| + \frac{x^{-1}}{(-1)} + C$
$= -x^2 - \ln|x| - x^{-1} + C$. We have used the formula $\int x^a dx = \frac{x^{a+1}}{a+1} + C$ if $a \neq -1$ and $\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + C$.

(b)

$$\int \frac{-2x^3 - x + 1}{\sqrt{x}} dx$$

Solution. We can first simplify
$\frac{-2x^3-x+1}{\sqrt{x}} = -2\frac{x^3}{x^{\frac{1}{2}}} - \frac{x}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}}} = -2x^{\frac{5}{2}} - x^{\frac{1}{2}} + x^{-\frac{1}{2}}$.
So $\int \frac{-2x^3-x+1}{\sqrt{x}}dx = \int(-2x^{\frac{5}{2}} - x^{\frac{1}{2}} + x^{-\frac{1}{2}})dx$
$= -2\int x^{\frac{5}{2}}dx - \int x^{\frac{1}{2}}dx + \int x^{-\frac{1}{2}}dx = -2\cdot\frac{x^{\frac{7}{2}}}{\frac{7}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$
$= -\frac{4}{7}x^{\frac{7}{2}} - \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

(c)

$$\int (x-2)(x+3)dx$$

Solution: First we simplify $(x-2)(x+3) = x(x+3) - 2(x+3) = x^2 + 3x - 2x - 6 = x^2 + x - 6$.
Thus $\int(x-2)(x+3)dx = \int(x^2 + x - 6)dx = \frac{x^3}{3} + \frac{x^2}{2} - 6x + C$.

(d)

$$\int -4\sec^2(\frac{x}{2}) - 3\cos(2x) - 4\sin(\frac{x}{3})dx$$

Solution: $\int -4\sec^2(\frac{x}{2}) - 3\cos(2x) - 4\sin(\frac{x}{3})dx$
$= -4\int \sec^2(\frac{x}{2})dx - 3\int\cos(2x)dx - 4\int\sin(\frac{x}{3})dx$
$= -4\cdot\frac{\tan(\frac{x}{2})}{\frac{1}{2}} - 3\frac{\sin(2x)}{2} - 4\frac{(-\cos(\frac{x}{3}))}{\frac{1}{3}}$

$= -8\tan(\frac{x}{2}) - \frac{3\sin(2x)}{2} + 12\cos(\frac{x}{3}) + C$. We have used $\int \sec^2(ax)dx = \frac{\tan(ax)}{a} + C$, $\int \cos(ax)dx = \frac{\sin(ax)}{a} + C$, $\int \sin(ax)dx = -\frac{\cos(ax)}{a} + C$.

(e)

$$\int 4\sec(\frac{x}{2})\tan(\frac{x}{2}) - 4e^{\frac{x}{3}}\,dx$$

Solution: $\int 4\sec(\frac{x}{2})\tan(\frac{x}{2}) - 4e^{\frac{x}{3}}\,dx = 4\int \sec(\frac{x}{2})\tan(\frac{x}{2})dx - 4\int e^{\frac{x}{3}}\,dx$
$= 4\cdot\frac{\sec(\frac{x}{2})}{\frac{1}{2}} - 4\cdot\frac{e^{\frac{x}{3}}}{\frac{1}{3}} + C = 8\sec(\frac{x}{2}) - 12e^{\frac{x}{3}} + C$.

We have used $\int \sec(ax)\tan(ax)dx = \frac{\sec(ax)}{a} + C$ and $\int e^{ax}dx = \frac{e^{ax}}{a} + C$.

(f)

$$\int \frac{1-x^2}{1+x^2} - \frac{3}{\sqrt{1-x^2}}dx$$

Solution: First, we simplify $\frac{1-x^2}{1+x^2} = \frac{2-1-x^2}{1+x^2} = \frac{2}{1+x^2} + \frac{-1-x^2}{1+x^2} = \frac{2}{1+x^2} - 1$.
Thus $\int \frac{1-x^2}{1+x^2} - \frac{3}{\sqrt{1-x^2}}dx = \int(\frac{2}{1+x^2} - 1)dx - 3\int \frac{1}{\sqrt{1-x^2}}dx$
$= 2\int \frac{1}{1+x^2}dx - \int 1dx - 3\arcsin(x) + C = 2\arctan(x) - x - 3\arcsin(x) + C$.
We have used $\int \frac{1}{1+x^2}dx = \arctan(x) + C$ and $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$.

(g)

$$\int 3^x - x^3dx$$

Solution: $\int 3^x dx - \int x^3dx = \frac{3^x}{\ln(3)} - \frac{x^4}{4} + C$. We have used $\int a^x dx = \frac{a^x}{\ln a} + C$.

2. Evaluate the following definite integrals: **Note that the functions in problem (a)-(c) are continuous on the given inteval. So we can use FTC to evaluate definite integrals.**

(a) $\int_1^4 \frac{-2x^3+1}{\sqrt{x}}dx$ Solution: First, we simplify the expression
$\frac{-2x^3+1}{\sqrt{x}} = \frac{-2x^3+1}{x^{\frac{1}{2}}} = -2\frac{x^3}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}}} = -2x^{3-\frac{1}{2}} + x^{-\frac{1}{2}} = -2x^{\frac{5}{2}} + x^{-\frac{1}{2}}$.
So $\int \frac{-2x^3+1}{\sqrt{x}}dx = \int(-2x^{\frac{5}{2}} + x^{-\frac{1}{2}})dx = -2\cdot\frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = -\frac{4}{7}x^{\frac{7}{2}} + 2x^{\frac{1}{2}} + C$.
This implies that
$\int_1^4 \frac{-2x^3+1}{\sqrt{x}}dx = -\frac{4}{7}x^{\frac{7}{2}} + 2x^{\frac{1}{2}}\Big|_1^4 = -\frac{4}{7}\cdot 4^{\frac{7}{2}} + 2\cdot 4^{\frac{1}{2}} - \Big(-\frac{4}{7}\cdot 1^{\frac{7}{2}} + 2\cdot 1^{\frac{1}{2}}\Big)$
Note that $4^{\frac{1}{2}} = \sqrt{4} = 2$, $4^{\frac{7}{2}} = (4^{\frac{1}{2}})^7 = 2^7 = 2*2*2*2*2*2*2 = 128$, $1^{\frac{5}{2}} = 1$ and $1^{\frac{1}{2}} = 1$.
So $-\frac{4}{7}\cdot 4^{\frac{7}{2}} + 2\cdot 4^{\frac{1}{2}} - \Big(-\frac{4}{7}\cdot 1^{\frac{7}{2}} + 2\cdot 1^{\frac{1}{2}}\Big) = -\frac{4}{7}\cdot 128 + 2\cdot 2 - \Big(-\frac{4}{7} + 2\Big) = -\frac{512}{7} + 4 + \frac{4}{7} - 2 = -\frac{508}{7} + 2 = \frac{-508+14}{7} = -\frac{494}{7}$.
Thus $\int_1^4 \frac{-2x^3+1}{\sqrt{x}}dx = -\frac{494}{7}$.

(b)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(\frac{x}{3})}{2} dx$$

Solution: We have $\int \frac{\cos(\frac{x}{3})}{2} dx = \frac{1}{2} \int \cos(\frac{x}{3}) dx = \frac{1}{2} \cdot \frac{\sin(\frac{x}{3})}{\frac{1}{3}} = \frac{3}{2}\sin(\frac{x}{3}) + C$. Thus $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(\frac{x}{3})}{2} dx = \frac{3}{2}\sin(\frac{x}{3})\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{3}{2}\sin(\frac{\frac{\pi}{2}}{3}) - \frac{3}{2}\sin(\frac{-\frac{\pi}{2}}{3}) = \frac{3}{2}\sin(\frac{\pi}{6}) - \frac{3}{2}\sin(-\frac{\pi}{6}) = \frac{3}{2} \cdot \frac{1}{2} - \frac{3}{2} \cdot (-\frac{1}{2}) = \frac{3}{4} + \frac{3}{4} = \frac{3}{2}$.

(c)

$$\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} 4\sin(3x) dx$$

Solution: $\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} 4\sin(3x) dx = -4\frac{\cos(3x)}{3}\Big|_{\frac{\pi}{18}}^{\frac{\pi}{9}} = -4 \cdot \frac{\cos(3 \cdot \frac{\pi}{9})}{3} - (-4\frac{\cos(3 \cdot \frac{\pi}{18})}{3}) = -\frac{4}{3}\cos(\frac{\pi}{3}) + \frac{4}{3}\cos(\frac{\pi}{6}) = -\frac{4}{3} \cdot \frac{1}{2} + \frac{4}{3} \cdot \frac{\sqrt{3}}{2} = -\frac{2}{3} + \frac{2\sqrt{3}}{3}$.. We have used $\cos(\frac{\pi}{3}) = \frac{1}{2}$ and $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$.

(d)

$$\int_{-1}^{1} \frac{1}{2x+1} dx$$

Note that the function $f(x) = \frac{1}{2x+1}$ is not continuous at $x = -\frac{1}{2}$. So $f(x) = \frac{1}{2x+1}$ is not continuous on the interval $[-1, 1]$. We can not use the FTC to evaluate $\int_{-1}^{1} \frac{1}{2x+1} dx$ for now.

(e)

$$\int_{0}^{2} \frac{1}{2x+1} dx$$

Solution:Note that the function $f(x) = \frac{1}{2x+1}$ is continuous on $[0, 2]$ $\int_0^2 \frac{1}{2x+1} dx = \frac{\ln|2x+1|}{2}\Big|_0^2 = \frac{\ln 5}{2} - \frac{\ln 1}{2} = \frac{\ln 5}{2}$. We have used $\int \frac{1}{ax+b} dx = \frac{\ln|ax+b|}{a} + C$ and $\ln 1 = 0$.

3. Suppose the value of the function f is shown in the following table

x	-1	-3/4	-1/2	-1/4	0	1/4	1/2	3/4	1
f(x)	1	-1	2	-2	3	0	-1	2	3

(a) Approximate $\int_0^1 f(x) dx$ using 4 equal subintervals and left endpoints.
Solution: The length of each subinterval is $\frac{1-0}{4} = \frac{1}{4}$. So the partition is $\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$. The Riemann sum is $f(0) \cdot \frac{1}{4} + f(\frac{1}{4}) \cdot \frac{1}{4} + f(\frac{1}{2}) \cdot \frac{1}{4} + f(\frac{3}{4}) \cdot \frac{1}{4} = 3 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + (-1) \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} = \frac{3}{4} - \frac{1}{4} + \frac{2}{4} = \frac{4}{4} = 1$.

(b) Approximate $\int_{-1}^{1} f(x)dx$ using 4 equal subintervals and left endpoints.
Solution: The length of each subinterval is $\frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2}$. So the partition is $\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$. The Riemann sum is $f(-1)\cdot\frac{1}{2} + f(-\frac{1}{2})\cdot\frac{1}{2} + f(0)\cdot\frac{1}{2} + f(\frac{1}{2})\cdot\frac{1}{2} = 1\cdot\frac{1}{2} + 2\cdot\frac{1}{2} + 3\cdot\frac{1}{2} + (-1)\cdot\frac{1}{2} = \frac{1}{2} + 1 + \frac{3}{2} - \frac{1}{2} = 2$.

4. Approximate $\int_{1}^{3}(4x^2-5)dx$ using 4 equal subintervals and left endpoints.
Solution:The length of each subinterval is $\frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$. So the partition is $\{1, \frac{3}{2}, 2, \frac{5}{2}, 3\}$. The Riemann sum is $f(1)\cdot\frac{1}{2} + f(\frac{3}{2})\cdot\frac{1}{2} + f(2)\cdot\frac{1}{2} + f(\frac{5}{2})\cdot\frac{1}{2}$. We have $f(1) = 4\cdot 1^2 - 5 = 4-5 = -1$, $f(\frac{3}{2}) = 4\cdot(\frac{3}{2})^2 - 5 = 4\cdot\frac{9}{4} - 5 = 9-5 = 4$, $f(2) = 4\cdot(2)^2 - 5 = 4\cdot 4 - 5 = 16-5 = 11$, $f(\frac{5}{2}) = 4\cdot(\frac{5}{2})^2 - 5 = 4\cdot\frac{25}{4} - 5 = 25-5 = 20$.
So $f(1)\cdot\frac{1}{2} + f(\frac{3}{2})\cdot\frac{1}{2} + f(2)\cdot\frac{1}{2} + f(\frac{5}{2})\cdot\frac{1}{2} == (-1)\cdot\frac{1}{2} + 4\cdot\frac{1}{2} + 11\cdot\frac{1}{2} + 20\cdot\frac{1}{2} = \frac{1}{2}(-1+4+11+20) = \frac{34}{2} = 17$.

5. Evaluate the following limits

(a)

$$\lim_{||P||\to 0}\sum_{k=1}^{n}(c_k^2-1)\Delta x_k$$

where $P = \{x_0 = 1, x_1, \cdots, x_k, \cdots, x_n = 2\}$, $c_k \in [x_{k-1}, x_k]$, $\Delta x_k = x_k - x_{k-1}$.
Solution: We have $f(c_k) = c_k^2 - 1$, $f(x) = x^2 - 1$, $a = 1$ and $b = 2$.
So $\lim_{||P||\to 0}\sum_{k=1}^{n}(c_k^2-1)\Delta x_k = \int_1^2(x^2-1)dx = \frac{x^3}{3} - x\Big|_1^2$
$= \frac{2^3}{3} - 2 - \left(\frac{1^3}{3} - 1\right) = \frac{2^3}{3} - 2 - \left(\frac{1^3}{3} - 1\right) = \frac{8}{3} - 2 - \frac{1}{3} + 1 = \frac{7}{3} - 1 = \frac{4}{3}$.

(b)

$$\lim_{||P||\to 0}\sum_{k=1}^{n}3\sin(2c_k)\Delta x_k$$

where $P = \{x_0 = 0, x_1, \cdots, x_k, \cdots, x_n = \frac{\pi}{4}\}$, $c_k \in [x_{k-1}, x_k]$, $\Delta x_k = x_k - x_{k-1}$.
Solution: We have $f(c_k) = 3\sin(2c_k)$, $f(x) = 3\sin(2x)$, $a = 0$ and $b = \frac{\pi}{4}$.
So $\lim_{||P||\to 0}\sum_{k=1}^{n}3\sin(2c_k)\Delta x_k = \int_0^{\frac{\pi}{4}}3\sin(2x)dx = -\frac{3\cos(2x)}{2}\Big|_0^{\frac{\pi}{4}}$
$= -\frac{3\cos(2\cdot\frac{\pi}{4})}{2} - \left(-\frac{3\cos(2\cdot 0)}{2}\right) = -\frac{3\cos(\frac{\pi}{2})}{2} + \frac{3\cos(0)}{2} = -\frac{3\cdot 0}{2} + \frac{3\cdot 1}{2} = \frac{3}{2}$. We have used $\cos(\frac{\pi}{2}) = 0$ and $\cos(0) = 1$.

6. Express the area of the region enclosed by $y = x^2 - 1$ and $y = 5x + 5$ as an definite integral (**Do not evaluate the definite integral**).

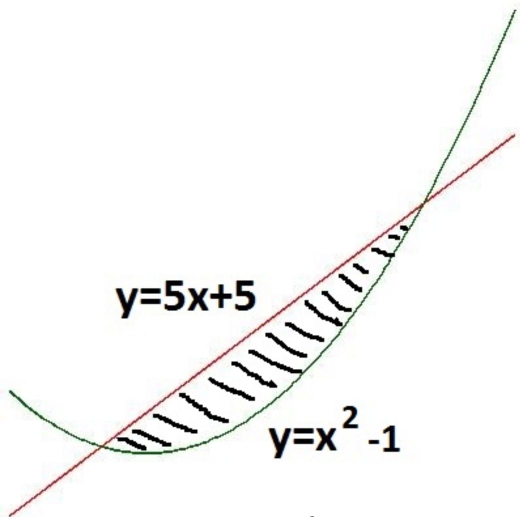

FIGURE 1. Graph for problem 6

Solution: We first find the points of intersection by solving $x^2 - 1 = 5x + 5$. This implies that $x^2 - 1 - 5x - 5 = 0$ and $x^2 - 5x - 6 = 0$. We get $(x-6)(x+1) = 0$ and $x = -1$ or $x = 6$.

From the graph, we know that $5x + 5 \geq x^2 - 1$ on the interval $[-1, 6]$. So the area of the region enclosed by $y = x^2 - 1$ and $y = 5x + 5$ is

$$\int_{-1}^{6} 5x + 5 - (x^2 - 1)dx = \int_{-1}^{6} (5x + 5 - x^2 + 1)dx = \int_{-1}^{6} (-x^2 + 5x + 6)dx.$$

7. **(a)** Express the area of the region enclosed by $y = -\sqrt{x+1}$, $y = -2x + 4$, $x-$axis and $y-$axis as an definite integral.
Solution: We need to find the x coordinates of these points on the graph. The x coordinate of point 1 is $x = 0$.
The x coordinate of point 2 is given by $-2x + 4 = 0$ which gives $2x = 4$ and we get $x = 2$. The x coordinate of point 3 is given by solving $-2x + 4 = -\sqrt{x+1}$ which gives $(-2x+4)^2 = (-\sqrt{x+1})^2$ and we get $4x^2 - 16x + 16 = x + 1$, $4x^2 - 16x - x + 16 - 1 = 0$ and $4x^2 - 17x + 15 = 0$. Factoring $4x^2 - 17x + 15 = 0$, we get $(x-3)(4x-5) = 0$ and $x = 3$ or $x = \frac{5}{4}$.
Plugging $x = 3$ into $-2x + 4$, we get $-6 + 4 = -2$.
Plugging $x = 3$ into $-\sqrt{x+1}$, we get $-\sqrt{3+1} = -2$.
So $x = 3$ is the solution of $-2x + 4 = -\sqrt{x+1}$.
Plugging $x = \frac{5}{4}$ into $-2x + 4$, we get $-2 \cdot \frac{5}{4} + 4 = -\frac{5}{2} + 4 = \frac{3}{2}$.
Plugging $x = \frac{5}{4}$ into $-\sqrt{x+1}$, we get $-\sqrt{\frac{5}{4} + 1} = -\sqrt{\frac{9}{4}} = -\frac{3}{2}$.

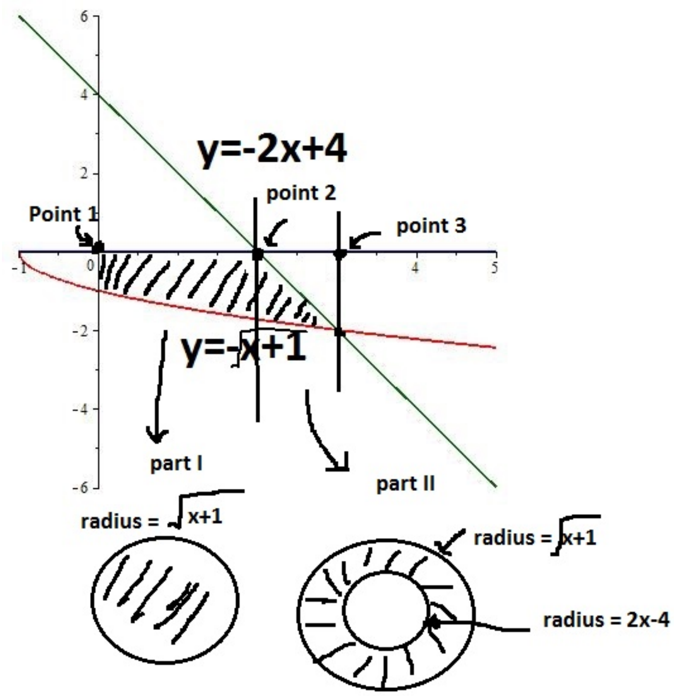

FIGURE 2. Graph for problem 7

So $x=\frac{5}{4}$ is not the solution of $-2x+4=-\sqrt{x+1}$.
So the x coordinate of point 3 is $x=3$.
On the interval $[0,2]$, we have $0\geq -\sqrt{x+1}$ and the area of the region bounded by $y=0$, $y=-\sqrt{x+1}$, $x=0$ and $x=2$ is $\int_0^2 0-(-\sqrt{x+1})dx=\int_0^2\sqrt{x+1}dx$.
On the interval $[2,3]$, we have $-2x+4\geq -\sqrt{x+1}$ and the area of the region bounded by $y=-2x+4$, $y=-\sqrt{x+1}$, $x=2$ and $x=3$ is
$\int_2^3 -2x+4-(-\sqrt{x+1})dx=\int_2^3\Big(-2x+4+\sqrt{x+1}\Big)dx$.

So the area of the shaded region is
$\int_0^2\sqrt{x+1}dx+\int_2^3\Big(-2x+4+\sqrt{x+1}\Big)dx$.

(b) Express the volume of the solid obtained by rotating the above enclosed region about $x-axis$ as an definite integral.

Solution: On the interval $[0,2]$, the cross section is a circle of radius $\sqrt{x+1}$ and $A(x)=\pi(\sqrt{x+1})^2=\pi(x+1)$. The volume of the first part is $\int_0^2\pi(x+1)dx$. On the interval $[2,3]$, the cross section is an annulus where the bigger circle has radius $-(-\sqrt{x+1})=\sqrt{x+1}$ and the smaller circle has radius $-(-2x+4)=2x-4$. So the area of the cross section is $A(x) = \pi(\sqrt{x+1})^2 - \pi(2x-4)^2 = \pi(x+1-(4x^2-16x+16)) = \pi(-4x^2+17x-15)$, The volume of the second part is $\int_2^3 \pi(-4x^2+17x-15)dx$. Thus the volume we are interested in finding is $\int_0^2 \pi(x+1)dx + \int_2^3 \pi(-4x^2+17x-15)dx$.

8. Express the area of the region enclosed by $y = -\sqrt{x}$, $y = x$ and $x = 4$ as an definite integral. Solution: From the picture, we see that $x \geq -\sqrt{x}$

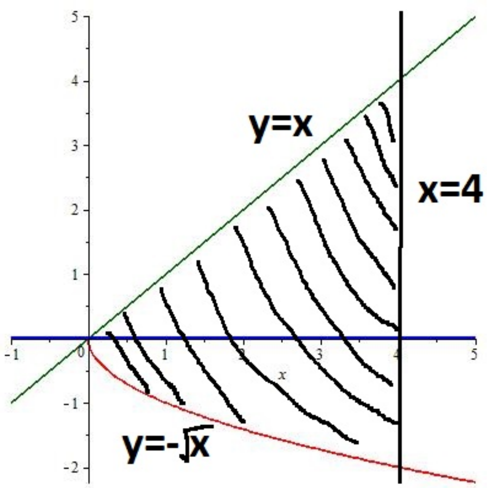

FIGURE 3. Graph for problem 8

on the interval $[0, 4]$. So the area of the region enclosed by $y = -\sqrt{x}$, $y = x$ and $x = 4$ is

$$\int_0^4 x - (-\sqrt{x})dx = \int_0^4 (x + \sqrt{x})dx.$$

9. **(a)** Express the area of the region enclosed by $y = 4x$, $y = 2x^2$ as an definite integral(**Do not evaluate the definite integral**).
Solution: First, we find the points where $y = 4x$ and $y = 2x^2$ intersect by solving $4x = 2x^2$. We get $4x - 2x^2 = 2x(2-x) = 0$ and $x = 0$ or $x = 2$. On the interval, we have $4x \geq 2x^2$. So the area of the region enclosed by $y = 4x$, $y = 2x^2$ is $\int_0^2 (4x - 2x^2)dx$.

(b) Express the volume of the solid obtained by rotating the above enclosed region about $x-axis$ as an definite integral(**Do not evaluate the definite integral**).
Solution: The cross section is an annulus where the radius of the bigger circle is $4x$ and the radius of the smaller circle is $2x^2$. So the area of the cross section is $\pi \cdot (4x)^2 - \pi \cdot (2x^2)^2 = \pi \cdot 16x^2 - \pi \cdot 4x^4 = \pi(16x^2 - 4x^4)$.

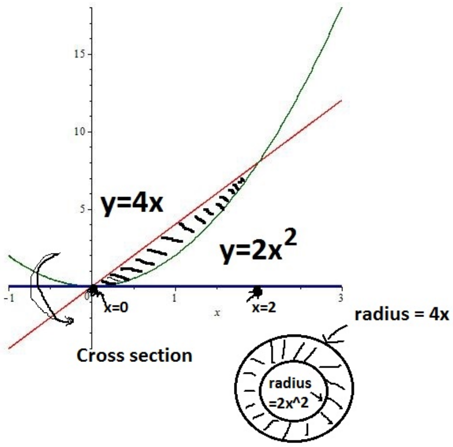

FIGURE 4. Graph for problem 9

So the volume of the solid obtained by rotating the above enclosed region about $x - axis$ is $\int_0^2 \pi(16x^2 - 4x^4)dx$