Problem Set #5

Due: Wednesday, Feb. 15

1. Let $D^2 = \{(x, y) \in R^2 | \ x^2 + y^2 \leq 1\}$. Prove that the closed unit disk D^2 in $\mathbb{R}^2$ cannot be retracted to the unit circle S^1.

2. Use previous problem to deduce that any continuous map $f : D \mapsto D$ has a fixed point. (*Hint:* Prove it by contradiction. Consider the line joining x to $f(x)$ where $x \in D$.)

3. **(a)** The space G is a topological group meaning that G is a group and also a Hausdorff topological space such that the multiplication and map taking each element to its inverse are continuous operations. Given two loops based at the identity e in G, say $\alpha(s)$ and $\beta(s)$, we have two ways to combine them: $\alpha \cdot \beta$ (product of loops as in the definition of fundamental group) and secondly $\alpha\beta$ using the group multiplication. Show, however, that these constructions give homotopic loops.

(b) Show that the fundamental group of a path-connected topological group is abelian. (Hint: Show that $\alpha\beta \sim \beta\alpha$.You may want to use the hint at problem 8-3 on page 191 of the textbook.)

4. Show that the following are equivalent:
(a) X is contractible.
(b) X is homotopy equivalent to a one-point space.
(c) Any point of X is a deformation retract of X.

5. Determine the fundamental group of the Möbius band.