1. Let $\phi : S^1 \to S^1$ be a continuous map such that $\phi(1) = 1$. (Here we use the complex number notation. We identify $S^1$ as the image of $e^{i\theta}$.) Because $\pi_1(S^1, 1)$ is infinite cyclic, there is an integer $n$, called the degree of $\phi$ and denoted by $\deg(\phi)$, such that $\phi(\gamma) = \gamma^n$ for all $\gamma \in \pi_1(S^1, 1)$. If $\phi : S^1 \to S^1$ is an arbitrary continuous map, we define the degree of $\phi$ to be the degree of $\frac{\phi}{\phi(1)}$ (in complex notation), which takes 1 to 1.

(a) Show that two maps $\phi, \psi : S^1 \to S^1$ are homotopic if and only if they have the same degree.

(b) Show that $\deg(\phi \circ \psi) = \deg(\phi) \cdot \deg(\psi)$ for any two continuous maps $\phi, \psi : S^1 \to S^1$.

(c) For each $n \in \mathbb{Z}$, compute the degrees of the $n$th power map $p_n(z) = z^n$ and its conjugate $\bar{p}_n(z) = \bar{z}^n$.

(d) Show that $\phi : S^1 \to S^1$ has an extension to a continuous map $\Phi : D^2 \to S^1$ if and only if it has degree zero.

2. Prove the fundamental theorem of algebra: Every complex polynomial of positive degree has a zero. [Hint: If $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, write $p_\varepsilon(z) = \varepsilon p(\varepsilon)$ and show that there exists $\varepsilon > 0$ such that $|p_\varepsilon(z) - z^n| < 1$ when $z \in S^1$. If $p$ has no zeros, prove that $p_\varepsilon|_{S^1}$ is homotopic to $p_n(z) = z^n$, and use the results of previous problem to derive a contradiction.]

3. Consider the group of $2 \times 2$ real matrices with nonzero determinant

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$ 

By identifying these matrices with a subset of $\mathbb{R}^4$ using the four entries, we can make this into a topological space. This space is denoted $GL(2, \mathbb{R})$ and is called the general linear group of $2 \times 2$ real matrices. Let $O(2) \subset GL(2, \mathbb{R})$ be the subgroup of the orthogonal matrices $Q$ satisfying $QQ^t = Q^tQ = I$.

(a) Use the Gram-Schmidt orthogonalization process which writes a given element $A$ in $GL(2, \mathbb{R})$ uniquely as a product $A = QR$, where $Q \in O(2)$ and $R$ is an upper triangular matrix with positive diagonal entries to show that there is a deformation retraction of $GL(2, \mathbb{R})$ onto $O(2)$.

(b) Let $SO(2)$ denote the matrices in $O(2)$ with determinant 1. Show that there is a homeomorphism (but not a group isomorphism) between $O(2)$ and the product space $SO(2) \times \mathbb{Z}_2$.

(c) Show that $SO(2)$ is homeomorphic to $S^1$, and this homeomorphism is a group homomorphism.

(d) Compute the fundamental groups $\pi_1(SO(2), I_2), \pi_1(O(2), I_2)$ and $\pi_1(GL(2, \mathbb{R}), I_2)$ where $I_2$ is the identity matrix.