Problem Set #6

Due: Wednesday, Feb. 22

1. Let $\phi : S^1 \mapsto S^1$ be a continuous map such that $\phi(1) = 1$. (Here we use the complex number notation. We identify S^1 as the image of $e^{i\theta}$.) Because $\pi_1(S^1, 1)$ is infinite cyclic, there is an integer n, called the degree of ϕ and denoted by $deg(\phi)$, such that $\phi(\gamma) = \gamma^n$ for all $\gamma \in \pi_1(S^1, 1)$. If $\phi : S^1 \mapsto S^1$ is an arbitrary continuous map, we define the degree of ϕ to be the degree of $\frac{\phi}{\phi(1)}$ (in complex notation), which takes 1 to 1.

- **(a)** Show that two maps ϕ, $\psi : S^1 \mapsto S^1$ are homotopic if and only if they have the same degree.
- **(b)** Show that $deg(\phi \circ \psi) = deg(\phi) \cdot deg(\psi)$ for any two continuous maps ϕ, $\psi : S^1 \mapsto S^1$.
- **(c)** For each $n \in Z$, compute the degrees of the nth power map $p_n(z) = z^n$ and its conjugate $\bar{p}_n(z) = \bar{z}^n$.
- **(d)** Show that $\phi : S^1 \mapsto S^1$ has an extension to a continuous map $\Phi : D^2 \mapsto S^1$ if and only if it has degree zero.

2. Prove the fundamental theorem of algebra: Every complex polynomial of positive degree has a zero. [Hint: If $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, write $p_\epsilon(z) = \epsilon^n p(\frac{z}{\epsilon})$ and show that there exists $\epsilon > 0$ such that $|p_\epsilon(z) - z^n| < 1$ when $z \in S^1$. If p has no zeros, prove that $p_\epsilon|_{S^1}$ is homotopic to $p_n(z) = z^n$, and use the results of previous problem to derive a contradiction.]

3. Consider the group of 2×2 real matrices with nonzero determinant

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

By identifying these matrices with a subset of $\mathbf{R}^4$ using the four entries, we can make this into a topological space. This space is denoted $GL(2, \mathbf{R})$ and is called the general linear group of 2×2 real matrices. Let $O(2) \subset GL(2, \mathbf{R})$ be the subgroup of the orthogonal matrices Q satisfying $QQ^t = Q^tQ = I$.

- **(a)** Use the Gram-Schmidt orthogonalization process which writes a given element A in $GL(2, \mathbf{R})$ uniquely as a product $A = QR$, where $Q \in O(2)$ and R is an upper triangular matrix with positive diagonal entries to show that there is a deformation retraction of $GL(2, \mathbf{R})$ onto $O(2)$.
- **(b)** Let $SO(2)$ denote the matrices in $O(2)$ with determinant 1. Show that there is a homeomorphism (but not a group isomorphism) between $O(2)$ and the prodcut space $SO(2) \times \mathbf{Z}_2$.
- **(c)** Show that $SO(2)$ is homeomorphic to S^1, and this homeomorphism is a group homomorphism.
- **(d)** Compute the fundamental groups $\pi_1(SO(2), I_2)$, $\pi_1(O(2), I_2)$ and $\pi_1(GL(2, \mathbf{R}), I_2)$ where I_2 is the identity matrix.