Calculus III Chapter 13

August 26, 2012

The Question

Given a space curve or even a plane curve what is its length? If we think of it as the path of a particle we are asking what is the distance traveled?

Procedure

Let us say

$$\mathbf{r}(t), a \leq t \leq b$$

is the vector function that describes the arc. Thinking of it as the path of a particle, $\frac{d\mathbf{r}}{dt}$ would be velocity and its length $|\frac{d\mathbf{r}}{dt}|$ would be the speed and the distance traveled would be

$$\int_a^b \left|\frac{d\mathbf{r}}{dt}\right| \, dt$$

the arclength. Put this way it boils down to integration of scalar function. This involves differentiation, calculating the length of a vector, and finally integrating a scalar valued function.

Exercise 3

Given $\mathbf{r}(t) = t\mathbf{i} + (2/3)t^{3/2}\mathbf{j}, 0 \leq t \leq 8$, find the unit tangent vector and length of the indicated portion.

Workout

Differentiation gives $\frac{d\mathbf{r}}{dt} = \mathbf{i} + t^{1/2}\mathbf{j}$ and its length would be $\sqrt{1^2 + (t^{1/2})^2} = (1+t)^{1/2}$. So arclength is

$$\int_0^8 (1+t)^{1/2}\, dt = (2/3)(1+t)^{3/2}\Big|_0^8 = 52/3.$$

The unit tangent vector would be

$$(1/\sqrt{1+t})\mathbf{i} + \sqrt{t/(1+t)}\mathbf{j}.$$

Exercise 5

Given $\mathbf{r}(t) = \cos^3 t\mathbf{i} + \sin^3 t\mathbf{j}, 0 \le t \le \pi/2$, find the unit tangent vector and length of the indicated portion.

Workout

Differentiation gives $\frac{d\mathbf{r}}{dt} = 3\cos^2 t(-\sin t)\mathbf{i} + 3\sin^2 t \cos t\mathbf{j}$ and its length would be $\sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} = 3\sin t \cos t$. So arclength is

$$\int_0^{\pi/2} 3\sin t \cos t \; dt = \; (3/2)\sin^2 t\Big|_0^{\pi/2} = 3/2.$$

The unit tangent vector would be

$$-\cos t\mathbf{i} + \sin t\mathbf{j}.$$

Exercise 9

Find the point on the curve

$$\mathbf{r}(t) = 5\sin t\mathbf{i} + 5\cos t\mathbf{j} + 12t\mathbf{k}$$

at a distance 26π units along the curve from the point $(0, 5, 0)$ in the direction of increasing arclength.

Workout

Differentiation gives $\frac{d\mathbf{r}}{dt} = 5\cos t\mathbf{i} - 5\sin t\mathbf{j} + 12\mathbf{k}$ and its length would be $\sqrt{25\cos^2 t + 25\sin^2 t + 144} = 13$. So arclength from $t = 0$ to $t = u$ is

$$\int_0^u 13\, dt = 13u.$$

The asked for point would be when $t = 2\pi$ and so it is $(0, 5, 24\pi)$.

Exercise 15

Find the length of the curve

$$\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + \sqrt{2}t\mathbf{j} + (1 - t^2)\mathbf{k}$$

from $(0, 0, 1)$ to $(\sqrt{2}, \sqrt{2}, 0)$.

Workout

Differentiation gives $\frac{d\mathbf{r}}{dt} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - 2t\mathbf{k}$ and its length would be $\sqrt{2+2+4t^2} = 2\sqrt{1+t^2}$. So arclength from $t = 0$ to $t = 1$ is

$$\int_0^1 2\sqrt{1+t^2}\,dt = 2(1/2)(t\sqrt{1+t^2} - \ln(t+\sqrt{1+t^2}))\Big|_0^1$$

$$= \sqrt{2} - \ln(1+\sqrt{2})$$

Length and parametrization

Does length depend on how the curve is parametrized? For example the following two curves

$$a)\mathbf{r}(t) = \cos 4t\mathbf{i} + \sin 4t\mathbf{j} + 4t\mathbf{k}, 0 \leq t \leq \pi/2$$

$$b)\mathbf{r}(t) = \cos(t/2)\mathbf{i} + \sin(t/2)\mathbf{j} + (t/2)\mathbf{k}, 0 \leq t \leq 4\pi$$

describe the same path. In the case of a) the speed is

$$\sqrt{16\sin^2 4t + 16\cos^2 4t + 16} = \sqrt{3}2 = 4\sqrt{2},$$

and so the distance traveled would be $4\sqrt{2}\pi/2 = 2\sqrt{2}\pi$. In the case of b) the speed is $\sqrt{(1/4)\sin^2(t/2) + (1/4)\cos^2(t/2) + 1/4} = \sqrt{(}1/2)$, and the distance traveled would be $(\sqrt{1/2})4\pi = 2\sqrt{2}\pi$.

A problem of golf

A golf ball is hit with an initial speed of 116 ft/sec at an angle of elevation 45 degrees from the tee to a green that is elevated 45 ft above the tee. Assuming the pin is 369 ft downrange, does not get in the way, where will the ball land in relation to the pin?

Computing the trajectory of the ball

Equation of motion for the ball moving under the sole force of gravity is

$$\frac{d^2\vec{r}}{dt^2} = -g\vec{j}.$$

This is the same for all projectiles. Integrating twice we get

$$\vec{r}(t) = \vec{a} + t\vec{b} - \frac{gt^2}{2}\vec{j},$$

where $\vec{a}$ is the initial position, $\vec{b}$ is the initial velocity.

Computing the trajectory of the ball

Equation of motion for the ball moving under the sole force of gravity is

$$\frac{d^2\vec{r}}{dt^2} = -g\vec{j}.$$

This is the same for all projectiles. Integrating twice we get

$$\vec{r}(t) = \vec{a} + t\vec{b} - \frac{gt^2}{2}\vec{j},$$

where $\vec{a}$ is the initial position, $\vec{b}$ is the initial velocity.
In our case $\vec{a} = \vec{0}, \vec{b} = 116\cos(\pi/4)\vec{i} + 116\sin(\pi/4)\vec{j}$.

Computing the trajectory of the ball

Equation of motion for the ball moving under the sole force of gravity is

$$\frac{d^2\vec{r}}{dt^2} = -g\vec{j}.$$

This is the same for all projectiles. Integrating twice we get

$$\vec{r}(t) = \vec{a} + t\vec{b} - \frac{gt^2}{2}\vec{j},$$

where $\vec{a}$ is the initial position, $\vec{b}$ is the initial velocity.
In our case $\vec{a} = \vec{0}, \vec{b} = 116\cos(\pi/4)\vec{i} + 116\sin(\pi/4)\vec{j}$.
Hence we get

$$x(t) = \frac{116t}{\sqrt{2}}, y(t) = \frac{116t}{\sqrt{2}} - \frac{gt^2}{2}.$$

To answer the question we need to solve the quadratic equation $y(t) = 45$ and if the solutions are t_1, t_2, we evaluate $x(t_1), x(t_2)$ and compare it with 369.

Solutions of the quadratic equation are 4.50, .625 and the corresponding values of $x(t)$ are 369.255, 51.245. So it reaches the pin within 0.255 ft i.e., 3 inches. Pretty good eh?

Arclength parameter and the Unit Tangent vector

Arclength parameter and the Unit Tangent vector

Assume λ is one of those zeros and $B(\lambda) = \delta$. Let us look at the inverse image of δ. It has at most 3 distinct points.

Inverse image of δ

If there is a point μ other than λ in the inverse image, it cannot be a zero of B'. Because if it were, by taking into account multiplicity inverse images of points close to δ would contain at least 4 points a contradiction, our Blaschke product being of degree 3. Hence we can throw away λ from the domain yet keep δ in the image.

If Not

Then the only remaining possibility is that λ is a triple root of

$$B(z) = \delta,$$

which means

$$(z^3 + \alpha z - \beta) - \delta(1 + \alpha z^2 - \beta z^3) = 0$$

has three equal roots and so

$$(1 + \delta\beta)z^3 - \delta\alpha z^2 + \alpha z - \beta - \delta = 0$$

has three equal roots.

Three Equal Roots

$$(1+\delta\beta)z^3 - \delta\alpha z^2 + \alpha z - \beta - \delta = 0$$

has three equal roots. This cannot happen unless we have

$$3(1+\delta\beta)(\alpha) = \delta^2\alpha^2$$

and

$$3\delta\alpha(\beta+\delta) = \alpha^2.$$

Three Equal Roots

$$(1+\delta\beta)z^3 - \delta\alpha z^2 + \alpha z - \beta - \delta = 0$$

has three equal roots. This cannot happen unless we have

$$3(1+\delta\beta)(\alpha) = \delta^2\alpha^2$$

and

$$3\delta\alpha(\beta+\delta) = \alpha^2.$$

Rewriting the identities we get

$$\delta^2\alpha^2 - 3\delta\beta\alpha - 3\alpha = 0$$

and

$$3\delta^2\alpha + 3\delta\alpha\beta - \alpha^2 = 0.$$

Adding the last two identities we have

$$(\delta^2 - 1)(3\alpha + \alpha^2) = 0.$$

Final Stretch

δ is in the open disk and so $\delta^2 - 1 \neq 0$ and therefore either $\alpha = 0$ or $\alpha = -3$. But $|\alpha| = |ab + bc + ca| < 3$ and so $\alpha = 0$. Now we have three reals a, b, c satisfying $a + b + c = 0, ab + bc + ca = 0$ which imply $a^2 + b^2 + c^2 = (a + b + c)^2 - 2\alpha = 0$. But then $a = b = c = 0$ a contradiction.

Yet Questions remain!

Are there examples of degree 4? Even more interesting is the question: What hyperbolic Riemann Surfaces admit locally conformal map onto the unit disk?

Cutting and Pasting

Brück's example of cutting and pasting.

The Plane

Let us return to the plane case. As noted earlier any such map must be constant plus $F(z) = \int_0^z e^{g(\zeta)} d\zeta$, for some entire function g. For this not to be surjective it should miss a value and that means F should miss a value say b. Hence $F(z) - b = e^{h(z)}$ where h is some entire function. To characterize locally conformal epimorphisms this way is not satisfactory. If we assume g is of finite order, we arrive at a necessary and sufficient condition for g for this to happen, namely g is a polynomial of degree bigger than 1.

Dr. Brück

In all honesty I must admit that he proved much more than hinted at here. He deals with functions of infinite order and makes an attempt to characterize all locally conformal maps of plane onto itself. As far as the disk goes he leaves us the job of finding ”all” such. Nevertheless it must be said the existence question is resolved satisfactorily as it was like a homework problem but complete classification is far from done. It is not even clear what it means!