PSS PROBLEMS IV

1. (MM 1766, April 2007) Let f be **differentiable** on $(0,\infty)$ and ω be a positive real number. Prove that if $\lim_{x\to\infty}(f'(x)+\omega f(x)) = A$, then $\lim_{x\to\infty} f(x) = A/\omega$.
   [Answer: Without lass of generality we can assume $A = 0$. Choose any $\epsilon > 0$. By hypothesis there exists a ξ large such that for $x \geq \xi$, $f'(x)+\omega f(x) \leq \epsilon$. Multiply by $e^{\omega x}$ and integrate from ξ to t and obtain

   $$e^{\omega t} f(t) - e^{\omega \xi} f(\xi) \leq \epsilon(e^{\omega t} - e^{\omega \xi})/\omega.$$

   Now divide both sides by $e^{\omega t}$ and let $t \to \infty$ to obtain $\limsup_{t\to\infty} f(t) \leq \epsilon/\omega$. Hence follows $\limsup_{t\to\infty} f(t) \leq 0$. Replacing $-f$ in place of f, we get that $\liminf_{t\to\infty} f(t) \geq 0$. QED. The above proof assumes f' is integrable. Do we really need that?]

2. Prove that $x^2 \geq (1+x)(\ln(1+x))^2$ for all $x > -1$.
   [Answer: Let $t = \ln(1+x)$. The inequality can be rewritten as

   $$(e^t - 1)^2 \geq e^t t^2$$

   for all t. Divide both sides by e^t and notice that the inequality takes the form

   $$4\sinh^2(t/2) \geq t^2$$

   for all t. Let $u = t/2$. Then we need only show that $\sinh^2 u \geq u^2$ or $\sinh u \geq u$ for $u > 0$. But by the mean value theorem $\sinh u = u \cosh v$ for some $0 < v < u$ and since $\cosh v \geq 1$ always, QED.]

3. (Crux 3286, November 2007) Is it possible to find a function $f : [0,1] \to \mathbb{R}$ such that

   $$f(x) = 1 + x\int_0^1 f(t)dt + x^2 \int_0^1 [f(t)]^2 dt?$$

   [Answer: No. Let $A = \int_0^1 f(t)dt, B = \int_0^1 [f(t)]^2 dt$. Integrating gives us that

   $$A = 1 + (A/2) + (B/3), 2B - 3A + 6 = 0.$$

This translates to $\int_0^1 2f(t)^2 - 3f(t) + 6 \; dt = 0$. But the integrand can be written as

$$2(f(t) - 3/4)^2 + (6 - 9/8)$$

which is always bigger than $6 - 9/8$ and so the integral cannot be zero. Contradiction. QED]

4. Suppose α, β, x, y are integers and $x \neq y$. Further $x - y$ divides $\alpha x + \beta y$. Show that $x - y$ divides $(\alpha + \beta)(x + y)$. [Apology for calling this a silly problem.]
[Answer: Let $d = x - y$. Obviously

$$\alpha x \equiv \alpha(x + y)(\text{mod d})$$

$$\beta y \equiv \beta(x + y)(\text{mod d}).$$

Hence

$$\alpha x + \beta y \equiv (\alpha + \beta)(x + y)(\text{mod d})$$

But by hypothesis

$$\alpha x + \beta y \equiv 0(\text{mod d})$$

and hence what is required.]

5. (MM 1767, 2007) Let G be the centroid of the triangle ABC. Prove that if the $\angle BAC = 60$ and $\angle BGC = 120$, then the triangle is equilateral.
[Answer: Let M be the midpoint of BC. Extend AM to AD so that $GM = MD$. Then $BGCD$ is a parallellogram and the $\angle BDC = 120$. So the quadrilateral $ABDC$ is cyclic. Assume the perpendicular bisector of BC meets the circum-circle of ABC in A'. Then $A'BC$ is equilateral because it is isosceles and the angle at A' is 60 degrees. Extend $A'M$ to meet the circle ABC in D'. Then we have $AM.MD = A'M.MD'$ but $AM : MD = A'M : MD' = 3 : 1$. Therefore $MD = MD'$. Therefor $D = D'$. QED