PSS PROBLEMS V
Furdui & Zelator

1. For three non-negative real numbers x, y, z satisfying the condition

$$xy + yz + zx = 3$$

show that

$$x^2 + y^2 + z^2 + 3xyz \geq 6.$$

[Answer: Proof by contradiction. Suppose not. The point where the last expression is less than 6 cannot be on the coordinate planes, nor for points above the planes $x = 4, y = 4, z = 4$. Thus a minimum must be achieved and it is less than 6 and it must be in the first quadrant. Applying the Lagrange Multipliers we obtain

$$2x+3yz+\lambda(y+z) = 0, 2y+3xz+\lambda(x+z) = 0, 2z+3xy+\lambda(x+y) = 0.$$

By playing these equations against each other we get the following:

$$2x-2y+3z(y-x)+\lambda(y-x) = 0, 2y-2z+3x(z-y)+\lambda(z-y) = 0,$$

$$2z - 2x + 3y(x - z) + \lambda(x - z) = 0.$$

If x, y, z are all different from each other, by cancelation we obtain that they are all equal, a contradiction. Thus we may assume that $x = y$ and $z \neq x$ and y. Now we restate the problem with only two variables x, z nonnegative real numbers subject to the constraint $x^2 + 2xz = 3$. We must show then that

$$g = 2x^2 + z^2 + 3x^2z \geq 6.$$

Substitute in g, $\dfrac{3 - x^2}{2x}$ for z to obtain

$$g = 2x^2 + \left(\frac{3 - x^2}{2x}\right)^2 + 3x^2\left(\frac{3 - x^2}{2x}\right)$$

$$= 6 + 2(x^2 - 3) + \left(\frac{3 - x^2}{2x}\right)^2 + 3x^2\left(\frac{3 - x^2}{2x}\right)$$

$$= 6 + \frac{(3-x^2)}{(2x)^2}\left[-2(2x)^2 + 3 - x^2 + \frac{3x}{2}(2x)^2\right]$$

$$= 6 + \frac{(3-x^2)}{(2x)^2}(6x^3 - 5x^2 + 3).$$

It is easy to check that $6x^3 - 5x + 3$ is positive for $x > 0$ and since $3 - x^2 > 0$ in our case, we see that $g \geq 6$. $\star\star$

2. Prove that $g_n = 2n(n+1)$ divides $F(m,n) = (n+1)^{2m} + n^{2m} - 1$ for all $n, m \in \mathbb{N}$.
[Answer: Proof by induction. For $m = 1$, $F(1,n) = (n+1)^2 + n^2 - 1 = 2n^2 + 2n = g_n$. Assume $m > 1$ and g_n divides $F(m-1,n)$. Write

$$F(m,n) = (n+1)^2(n+1)^{2m-2} + n^2 n^{2m-2} - 1$$

$$= F(m-1,n) + ((n+1)^2 - 1)(n+1)^{2m-2} + (n^2-1)n^{2m-2}$$

$$= F(m-1,n) + ((n+1)^2 + n^2 - 1)(n+1)^{2m-2} + (n^2-1)n^{2m-2} - n^2(n+1)^{2m-2}$$

$$= F(m-1,n) + (g_n(n+1)^{2m-2} + n(n+1)\left[(n-1)n^{2m-3} - n(n+1)^{2m-3}\right]$$

Since the number inside the last square brackets is always even, the theorem is obtained. $\star\star$

3. Let CH be the altitude of the triangle ABC with $\angle ACB = 90$. The internal bisector of $\angle BAC$ intersects CH, CB at P, M respectively. The internal bisector of $\angle ABC$ intersects CH, CA at Q, N respectively. Prove that the line passing through the mid points of PM and QN is parallel to AB.
[Answer: We first notice the distance of P, L form AB as $AP\sin(A/2), AL\sin(A/2)$ and also that $AP = AH\sec(A/2), AL = AC\sec(A/2)$. Hence the distance of the midpoint of PL from AB would be

$$(1/2)(AH + AC)\tan(A/2) = (1/2)(AC\cos(A) + AC)\tan(A/2)$$

$$= (1/2)(AB\cos^2(A) + AB\cos(A)\tan(A/2)$$

$$= (AB/2)\cos(A)(1 + \cos(A))\tan(A/2)$$

$$= (AB/2)\cos(A)2\cos^2(A/2))\tan(A/2) = (AB/2)\cos(A)\sin(A)$$

. By a similar argument we deduce that the distance of the midpoint of QN is $(AB/2)\cos(B)\sin(B)$. This proves what is required. $\star\star$

4. Compute

$$\lim_{n\to\infty} n \int_0^1 \frac{x^{kn}}{1+x^m}\, dx$$

where m, k fixed natural numbers.
[Answer: Change the variable to $t = x^m$ and obtain the integral to be

$$(1/m)\int_0^1 t^{(kn+1)/m-1}\frac{1}{1+t}\, dt.$$

Let $a = \frac{kn+1}{m}$ and evaluate the integral by integrating by parts twice to obtain the following:

$$\int_0^1 \frac{t^{a-1}}{1+t}\, dt = (1/2a) + (1/a)\int_0^1 \frac{t^a}{(1+t)^2}\, dt$$

$$= (1/2a) + (1/4a(a+1)) + (2/a(a+1))\int_0^1 \frac{t^{a+1}}{(1+t)^3}\, dt.$$

From this it is clear that

$$\lim_{a\to\infty} a\int_0^1 \frac{t^{a-1}}{1+t}\, dt = 1/2.$$

Hence

$$n\int_0^1 \frac{x^{kn}}{1+x^m}\, dx = (n/m)\int_0^1 \frac{t^{a-1}}{1+t}\, dt$$

and so the limit will be the same as the limit of

$$(n/m)(1/a)a\int_0^1 \frac{t^{a-1}}{1+t}\, dt$$

and that is the limit of $(n/m)(1/a)(1/2)$ and so it is $(1/2k)$.⋆⋆

5. Solve the trigonometric equation

$$\sin x + \cos x + \tan x + \csc x + \sec x + \cot x + 3 = 0.$$

[Answer: Let $t = \tan(x/2)$. Using Euler's identities we have

$$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + \frac{2t}{1-t^2} + \frac{1+t^2}{2t} + \frac{1+t^2}{1-t^2} + + \frac{1-t^2}{2t} + 3 = 0.$$

The fourth and the sixth term add up to $1/t$ and the third and the fifth term add to $(1+t)/(1-t)$. So the equation now simplifies to

$$3+\frac{1+2t-t^2}{1+t^2}+\frac{1+t}{1-t}+\frac{1}{t}=0.$$

Simplification leads to

$$\frac{t^4-4t-1}{(1+t^2)(t^2-t)}=0.$$

This has exactly two real roots.