

## PSS PROBLEMS VI

1. (B-1028, Fibonacci Quarterly, February 2007) Let  $\alpha = \frac{1 + \sqrt{5}}{2}$  and  $\beta = \frac{1 - \sqrt{5}}{2}$ . Prove that the **Diophantine** equation  $(a + b\alpha)^2 + (a + b\beta)^2 = c^2$  has no solutions in positive integers  $a, b, c$ .

[Answer: LHS of the equation can be simplified to

$$2a^2 + 2ab(\alpha + \beta) + b^2(\alpha^2 + \beta^2)$$

and we notice that  $\alpha + \beta = 1$  and  $\alpha^2 + \beta^2 = 3$ . Thus the equation can be written as

$$2a^2 + 2ab + 3b^2 = c^2.$$

Among all positive ordered triples  $(a, b, c)$  satisfying this equation if any there would exist one with a minimum  $a$ . Assume we have such a solution. We notice that if  $c$  is even, so are  $b$  and  $a$  and so the triple  $(a/2, b/2, c/2)$  is also a solution contradicting the minimality of  $a$ . So  $c$  and  $b$  are both odd. But then

$$c^2 \equiv b^2 \equiv 1 \pmod{8}.$$

Therefore the equation leads to

$$2a^2 + 2ab + 3b^2 = c^2 \equiv 1 \equiv 2a^2 + 2ab + 3 \pmod{8}.$$

Thus

$$2a^2 + 2ab \equiv -2 \pmod{8} \rightarrow a^2 + ab \equiv -1 \pmod{4}.$$

This means  $a(a + b)$  must be odd and so both  $a$  and  $a + b$  must be odd which is impossible since  $b$  is odd. QED]

2. (Crux, 3203, 2007) Let  $AB$  be the diameter of a semi-circle  $\Gamma$ . Let  $D$  be any point on the tangent to  $\Gamma$  at  $B$  lying on the same side of  $AB$  as  $\Gamma$ , let  $C$  be the midpoint of  $BD$ . the segments  $AC$  and  $AD$  intersect  $\Gamma$  for the second time at the points  $K$  and  $L$  respectively. If  $M$  and  $N$  are the projections of  $A$  and  $B$  onto  $KL$  respectively, show that  $ML = LK = KN$ .

[Answer: Let  $\phi$  and  $\psi$  denote  $\angle BAC$ ,  $\angle CAD$  respectively. The chord  $KL$  subtends the angle  $\psi$  at  $A$  and  $B$  and the  $\angle KBC = \phi$

on  $\Gamma$ . All this is standard with regard to circle in the plane. Note that the angles subtended by  $AB$  at  $K$  and  $L$  are right-angles. Now we see that  $\widehat{BKN}$  is equal the sum of the two internal angles  $\widehat{KBC} = \phi$  and  $\widehat{KLC} = \psi$  and so we obtain

$$\begin{aligned} KN &= BK \cos(\phi+\psi) = AB \sin(\psi) \cos(\phi+\psi) = BC \cot(\psi) \sin(\psi) \cos(\phi+\psi) \\ &= BC \cos(\psi) \cos(\phi + \psi). \end{aligned}$$

Also proceeding in the same manner, we obtain

$$\begin{aligned} LN &= BL \cos(\psi) = AB \sin(\phi+\psi) \cos(\psi) = BD \cot(\phi+\psi) \sin(\phi+\psi) \cos(\psi) \\ &= BD \cos(\psi) \cos(\phi + \psi). \end{aligned}$$

Since  $BD = 2BC$ , we have  $LN = 2KN$  or  $KN = KL$ .

Now we observe that

$$\begin{aligned} \widehat{KAM} &= 90 - \widehat{AKM} = 90 - (180 - \widehat{AKB} - \widehat{BKN}) \\ &= 90 - (180 - 90 - \widehat{BKN}) = \widehat{BKN} = \phi + \psi. \end{aligned}$$

Therefore

$$\begin{aligned} KM &= AK \sin(\phi+\psi) = AB \cos(\psi) \sin(\phi+\psi) = BD \cot(\phi+\psi) \cos(\psi) \sin(\phi+\psi) \\ &= BD \cos(\psi) \cos(\phi + \psi) = LN = 2KL. \end{aligned}$$

Hence  $ML = LK = KN$ . QED.] Zelator's idea: Drop a perpendicular from the center of  $\Gamma$  onto the chord  $KL$  and note that the foot is midpoint of both  $KL$  and  $MN$ . So it shows that  $KN = LM$  right away.

Vologodsky's idea is to compare similar triangles  $BKN$  and  $BAD$  and also another pair of similar triangles.

3. (CMJ 869, January 2008) **Problem of the week:** Prove that there exists no function  $f \in C^2([0, \infty))$  such that

$$f(x) > 0, f(x)f''(x) \leq -1 \text{ for all } x \geq 0.$$

[Answer: Proof by contradiction. Suppose  $f'(t) \leq 0$  for some  $t \in [0, \infty)$ . Since  $f''$  is negative,  $f'$  is strictly decreasing and there exists a  $\delta > 0$  and  $c > 0$  such that  $f'(x) \leq -c$  for  $x \geq t + \delta$  and  $f(x) - f(t + \delta) \leq -c(x - t - \delta)$  for all  $x > t + \delta$ .

This means  $f(x) < 0$  for all large  $x$ , a contradiction. Hence  $f'(x) > 0$  for all  $x$  in  $[0, \infty)$ . But then  $f(x) \geq f(0)$  for all  $x$  and  $f''(x) \leq -(1/f(x))$  for all  $x$ . That means

$$f'^2(x) - f'^2(0) = \int_0^x 2f'(u)f''(u)du \leq - \int_0^x 2f'(u)/f(u)du$$

which proves that  $f'^2(x) + 2\ln(f(x)) \leq f'^2(0) + 2\ln(f(0))$  and hence  $f(x)$  is bounded by  $c > 0$ . Hence  $f''(x) \leq -1/f(x) \leq -1/c$  for all  $x$  and

$$f'(x) - f'(0) = \int_0^x f''(u)du \leq -x/c$$

that implies  $f'(x) < 0$  eventually for all large  $x$ , a contradiction again. QED.]

4. (Mathematical Reflections, No.1,2008) The zeroes of the polynomial  $P(x) = x^3 + x^2 + ax + b$  are all real and negative. Prove that  $4a - 9b \leq 1$ .

[Answer: Let  $-p, -q, -r$  be the roots of the given polynomial. Then  $p + q + r = 1$ ,  $(pq + qr + rp) = a$ ,  $pqr = b$  and the function to maximize would be  $4a - 9b = 4(pq + qr + rp) - 9pqr$  subject to the constraints  $p + q + r = 1$ ,  $p, q, r$  all positive. Using the Lagrange multipliers we get

$$4q + 4r - 9qr + \lambda = 0, 4r + 4p - 9pr + \lambda = 0, 4p + 4q - 9pq + \lambda = 0.$$

Eliminating  $\lambda$  we get

$$4(q-p) - 9r(q-p) = 0, 4(r-q) - 9p(r-q) = 0, 4(p-r) - 9q(p-r) = 0.$$

If one of the  $4 - 9r, 4 - 9p, 4 - 9q$  is zero, we calculate  $4a - 9b$  and find it equals to  $80/81$  which is less than 1. If none of them are zero,  $p = q = r = 1/3$  and  $4a - 9b = 1$ . QED.]

5. Let  $f : [0, 1] \rightarrow [0, \infty)$  be a continuous function. Find the limit

$$\lim_{n \rightarrow \infty} \sqrt[n]{f\left(\frac{1}{n}\right)^n + f\left(\frac{2}{n}\right)^n + \dots + f\left(\frac{n}{n}\right)^n}.$$

[Answer: Let  $M = \max_{[0,1]} f(x)$ . Since  $\sqrt[n]{n}$  tends to 1 as  $n \rightarrow \infty$ , it is easy to see that the  $\limsup \leq M$ . If  $f$  is constantly equal

to  $M$ , the statement is obvious. Assume  $M > 0$  and choose  $\delta$  so that it is small, positive and less than  $M$ . Then there exists an open sub-interval  $J$  of  $[0, 1]$  on which  $f(x) > M - \delta$ . For any given positive integer  $n$ , let  $g(n)$  denote the number of fractions  $r/n$ ,  $r$  a positive integer  $\leq n$  that belong to  $J$ . It is easy to prove that  $g(n)/n$  tends to the length  $|J|$  of the interval  $J$ . By noting that

$$\sqrt[n]{f\left(\frac{1}{n}\right)^n + f\left(\frac{2}{n}\right)^n + \dots + f\left(\frac{n}{n}\right)^n} \geq \sqrt[n]{g(n)(M - \delta)^n}$$

and the lower function tends to the limit  $M - \delta$ , we get the  $\liminf \geq M - \delta$ . This being true for all sufficiently small positive  $\delta$ , we get

$$M \geq \limsup \geq \liminf \geq M.$$

QED.]