M.S. (Applied Mathematics) 
Comprehensive Examination in Analysis 

Do five (5) questions from each of Parts A and B.
Indicate on the front of the blue book which problems you wish to have graded.

R denotes the set of real numbers and C the set of complex numbers.

Part A. Real Analysis

1. Suppose \( \lim_{n \to \infty} a_n = A \) and \( \lim_{n \to \infty} b_n = B \). Prove that \( \lim_{n \to \infty} a_nb_n = AB \).

2. (a) Let \((M, d)\) be a metric space and suppose \( f_n : M \to \mathbb{R} \) are continuous functions. Define what it means for \( f_n \Rightarrow f \) uniformly on \( M \).
   (b) Prove: If \( f_n \Rightarrow f \) uniformly on \( M \) and each \( f_n \) is continuous on \( M \), then so is \( f \).

3. Abel’s Theorem: Suppose \( \sum_{k=0}^{\infty} a_k \) converges and let \( f_n(x) = \sum_{k=0}^{n} a_kx^k \). Show that \( f_n(x) \Rightarrow f(x) \) uniformly on \([0, 1]\), where \( f(x) = \sum_{k=0}^{\infty} a_kx^k \).

4. (a) Give two different but equivalent definitions of compactness for a metric space \((M, d)\).
    (b) Prove that a compact metric space is closed and bounded and give an example to show that the converse statement is not true.

5. Let \( f(x) \) be a continuous real-valued function on \([a, b]\) with \( f(x) \geq 0 \).
   Prove: If there is one point \( c \in [a, b] \) with \( f(c) > 0 \) then
   \[ \int_{a}^{b} f(x) \, dx > 0. \]

6. Let \( f_n(x) = x^n(1-x), \; g_n(x) = x^n(1-x^n), \; 0 \leq x \leq 1 \). Show that \( f_n(x) \Rightarrow 0 \) uniformly on \([0, 1]\) while \( g(x) \to 0 \) pointwise on \([0, 1]\) but not uniformly.

7. Let \( f : M \to \mathbb{R} \) be a continuous function on the compact metric space \((M, d)\).
   Prove that \( f \) is uniformly continuous.

8. (a) Define what it means for a subset \( O \) of \( M \) to be open, where \((M, d)\) is a metric space.
    (b) Prove that for any \( p \in M \) and \( r > 0 \), the ball \( B(p, r) = \{ q \in M : d(p, q) < r \} \) is an open subset of \( M \).
1. (a) Prove that if \( f(z) \) and \( \overline{f(z)} \) are analytic in a domain \( D \subseteq \mathbb{C} \) then \( f(z) \) is constant in \( D \). (Hint: Cauchy-Riemann equations)

(b) Prove that if \( f(z) \) is analytic in a domain \( D \subseteq \mathbb{C} \) and \(|f(z)|\) is constant in \( D \), then \( f(z) \) is constant in \( D \).

2. (a) Verify that the function \( u(x, y) = 2x(1 - y) \) is harmonic in \( \mathbb{R}^2 \) and find a harmonic conjugate \( v(x, y) \).

(b) Suppose \( u(x, y) \) and \( v(x, y) \) are conjugate harmonic functions on a domain \( D \subseteq \mathbb{R}^2 \). If \( U(x, y) = e^{u(x,y)} \cos v(x, y) \) and \( V(x, y) = e^{u(x,y)} \sin v(x, y) \), show that \( U(x, y) \) and \( V(x, y) \) are also conjugate harmonic.

3. (a) Evaluate the contour integral

\[
\int_C \frac{dz}{z}
\]

where \( C \) is the upper half of the circle \(|z| = 1\) from \( z = 1 \) to \( z = -1 \).

(b) Let \( C \) be the line segment from \( z = i \) to \( z = 1 \). Without evaluating the integral directly show that

\[
\left| \int_C \frac{dz}{z} \right| \leq 2
\]

4. Evaluate

\[
\oint_C \frac{\sin^2 z}{(z - \pi/6)^2}dz
\]

if \( C \) is the circle \(|z| = 1\) traced once counterclockwise.

5. (a) State Liouville’s Theorem.

(b) Suppose a non-constant function \( f(z) \) is such that, for two constants \( a > 0 \) and \( b > 0 \), \( f(z) = f(z + a) \) and \( f(z) = f(z + bi) \) for all \( z \in \mathbb{C} \). (Such a function is said to be doubly periodic.) Prove that \( f(z) \) is not analytic in the rectangle \( 0 \leq x \leq a, 0 \leq y \leq b \).

6. Compute all possible Laurent series for \( f(z) = \frac{1}{z^2 - 4z + 3} \) at \( z = 1 \). State explicitly the domain of convergence of each series.

7. Use residues to evaluate the contour integral

\[
\oint_C z(3z + 1)e^{2/z}dz
\]

where \( C = \{z : |z| = 1\} \) traced once counterclockwise.

8. Use residues to evaluate the improper integral

\[
\int_0^\infty \frac{1}{x^4 + 1}dz.
\]