M.S. Comprehensive Examination
Spring 2002

Instructions:

1. If you think that there is a mistake ask the proctor. If the proctor’s explanation is not satisfactory, interpret the problem as you see fit, but not in such a way that the answer is trivial.

2. From each part solve 3 of 5 problems.

3. If you solve more than three problems from a part, indicate the problems that you wish to have graded.

Part A

1. Consider the system

\[ \begin{align*}
\dot{x} &= \sin(t)y \\
\dot{y} &= -\cos(t)x
\end{align*} \]

Suppose a solution \( u(t) = (x(t), y(t)) \) has initial values \( u(0) = u_0 = (0, 1) \). Use the Fundamental Inequality to show that for \( 0 \leq t \leq \pi \)

\[ ||u(t) - u_0|| \leq \frac{1}{2}(1 - \sqrt{2} \cos(t - \pi/4)) \]

2. Consider the system \( \dot{x} = (1 - x^2)a(t) \) for a continuous function \( a(t) \). What condition on \( a(t) \) implies that the solution \( x(t) = 1 \) is Lyapunov stable. What condition implies that \( x(t) = 1 \) is asymptotically stable. Find an \( a(t) \) so that \( x(t) = 1 \) is uniformly asymptotically stable.

3. Consider the system

\[ \begin{align*}
\dot{x} &= -1 - x^2 + z^2 \\
\dot{y} &= 1 - y^2 + x^2 \\
\dot{z} &= 3 - 4xy - 3z^2.
\end{align*} \]

Find the stationary points and determine which are asymptotically stable.
4. Show that the unit circle is a limit cycle for the following equation

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
(1 - r^2)x & -r^2y \\
r^2x & + (1 - r^2)y
\end{bmatrix},
\]

where \( r = \sqrt{x^2 + y^2} \).

5. Find the fundamental solution of the system

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} = \begin{bmatrix}
3x & +4z \\
2x & +3z \\
-2x & +y & -2z
\end{bmatrix}.
\]

Part B

1.

(a) Solve the Cauchy problem \( u_x + \frac{1}{u} u_y = u^2 \) with the initial conditions \( u(x, 1) = 1 \)

(b) What condition on the initial data guarantees the existence of a solution in a neighborhood of the initial curve. Is this condition satisfied in this problem?

2. Find the canonical form and the general solution of the equation

\[
2xu_{xx} + 2(1 + xy)u_{xy} + 2yu_{yy} + \frac{2(1-x)}{1-xy}u_x + \frac{2(1-y)}{1-xy}u_y = 0.
\]

3.

(a) Find the solutions to the Dirichlet problem \( \triangle u + 5u = 0 \) and \( u_{\partial R} = 0 \) where \( R = \{(x, y)|0 < x < \pi, 0 < y < \pi\} \)

(b) What property of solutions to the Laplace equation on \( R \) is not shared with solutions to this equation. What feature of this equations causes this property to fail.
4. Let $A$ be a $2 \times 2$ matrix that has the real Jordan form \[
\begin{bmatrix}
c & 1 \\
0 & c
\end{bmatrix}.
\] For $u(x, y) = \begin{bmatrix} u_1(x, y) \\ u_2(x, y) \end{bmatrix}$ describe the general solution to the Cauchy-Kowalewski system

\[
\frac{\partial}{\partial x} u(x, y) = A \frac{\partial}{\partial y} u(x, y).
\]

5. Describe the symbol of the minimal surface equation

\[
(1 + u_y^2)u_{xx} - 2u_xu_yu_{xy} + (1 + u_x^2)u_{yy} = 0
\]

and show that it is elliptic.