1. If \( \{s_n\} \) is a complex sequence, define its arithmetic means by
\[
\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n + 1} \quad (n = 0, 1, 2, \ldots).
\]
If \( \lim s_n = s \), prove that \( \lim \sigma_n = s \).

2. Suppose that \( f \) is a real function defined on \( \mathbb{R}^1 \) which satisfies
\[
\lim_{h \to 0} [f(x + h) - f(x - h)] = 0
\]
for every \( x \in \mathbb{R}^1 \). Does this imply that \( f \) is continuous?

3. Suppose that \( f : X \to Y \) is a mapping between metric spaces \((X, d)\) and \((Y, \delta)\).
   
   (a) State the definition of uniform continuity of \( f \) in this setting.
   
   (b) Suppose that \( f \) is continuous and that \((X, d)\) is compact. Show that \( f \) is uniformly continuous.

4. Let \( X \) be an infinite set. For \( p, q \in X \), define
\[
d(p, q) = \begin{cases} 
1 & \text{if } p \neq q \\
0 & \text{if } p = q
\end{cases}
\]
Prove that this is a metric. What subsets of the resulting metric space are open? Which are closed? Which are compact?

5. Prove that the series
\[
\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}
\]
converges uniformly in every bounded interval, but does not converge absolutely for any value of \( x \).

6. Define
\[
f(x) = \begin{cases} 
e^{-1/\sqrt{x}} & \text{if } x \neq 0 \\
0 & \text{if } x = 0
\end{cases}
\]
Prove that \( f \) has derivatives of all orders at \( x = 0 \), and that \( f^{(n)}(0) = 0 \) for \( n = 1, 2, 3, \ldots \).
7. Let $f : [0, 1] \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 
0 & \text{if } x \text{ is rational} \\
x & \text{if } x \text{ is not rational}
\end{cases}$$

Determine whether or not $f$ is Riemann integrable on $[0,1]$. If it is then evaluate $\int_0^1 f(x) \, dx$.

**Part B: Complex Analysis**

Instructions: Do at least 2 questions from Part B

1. Compute all possible Laurent series at $z = 0$ for the function.

$$f(z) = \frac{1}{z^2 - z - 2}$$

Specify the domain of convergence of each series.

2. Let $u(x, y) = x^3 + 2xy - 3xy^2$.

   (a) Show that $u$ is harmonic.
   (b) Find all harmonic conjugates of $u$.
   (c) Find an analytic function $f(z)$ so that $u(x, y) = \Re f(x + iy)$.

3. Use the residue theorem to evaluate $\int_0^\infty \frac{1}{(x^2 + 4)^2} \, dx$