Part 1. Real Analysis. 100% will be obtained for complete answers to three questions. Indicate clearly which three questions you wish to be graded.

1. Use the definition of Cauchy sequence to prove that if \( \{x_n\} \) and \( \{y_n\} \) are Cauchy sequences then \( \{2x_n + y_n\} \) is a Cauchy sequence.

2. Suppose that the function \( f : \mathbb{R} \to \mathbb{R} \) has limit \( L \) at 0, and let \( a > 0 \). If \( g : \mathbb{R} \to \mathbb{R} \) is defined by \( g(x) = f(ax) \) for \( x \in \mathbb{R} \), show that \( \lim_{x \to 0} g(x) = L \).

3. Use the definition to show that if \( f(x) \) is Riemann integrable on \([a, b]\) and \( f(x) \) is Riemann integrable on \([b, c]\) then \( f(x) \) is Riemann integrable on \([a, c]\).

4. (a) In each of the following, determine if the given series converges. Explain your answer.
\[
(1) \sum_{n=1}^{\infty} \frac{n^4}{n!} \\
(2) \sum_{n=2}^{\infty} \frac{n}{\sqrt{n^4 - 2}}
\]

(b) Find the radius of convergence and the interval of convergence for the following series
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n2^n} x^n
\]
Part 2. Complex Analysis. 100% will be obtained for complete answers to three questions. Indicate clearly which three questions you wish to be graded.

1. Evaluate the Cauchy principal value of

\[ \int_0^\infty \frac{x \sin x}{x^2 + 9} \, dx \]

2. Expand \( f(z) = \frac{1}{(z - 1)(z - 3)^2} \) in a Laurent series valid for:

(a) \( 0 < |z - 1| < 2 \)

(b) \( 0 < |z - 3| < 2 \)

3. Find the image of the following curves under the reciprocal mapping \( w = \frac{1}{z} \). Draw the graphs.

(a) the semicircle \( |z| = 2, 0 \leq \arg z \leq \pi \)

(b) the line \( x = 1 \)

4. Find where the following functions are analytic.

(a) \( f(z) = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} \)

(b) \( f(z) = 3x^2y - 6ix^2y^2 \)