Part 1. Real Analysis. Answer 4 of the 6 questions in Part 1. If you answer more questions then indicate which you wish to be considered.

(1) Suppose that \( f \) is Riemann integrable on \([a, b]\) and \( g = f \) except at one point \( c \in [a, b] \). Prove that \( g \) is Riemann integrable on \([a, b]\) and
\[
\int_a^b f(x) \, dx = \int_a^b g(x) \, dx
\]

(2) (a) Define continuity for \( f : [a, b] \to \mathbb{R} \).
(b) Suppose that \( f, g \) are continuous on \([a, b]\) and define
\[
h(x) = \min \{ f(x), g(x) \}
\]
\[
H(x) = \max \{ f(x), g(x) \}
\]
for all \( x \in [a, b] \). Show that \( h \) and \( H \) are continuous on \([a, b]\).

(3) (a) Define the convergence of a sequence in a metric space \((X, d)\).
(b) Show that every convergent sequence of real numbers has a monotone subsequence that converges to the same limit.

(4) Suppose that \( f : [1, \infty) \to \mathbb{R} \) is a continuous function and \( \lim_{x \to \infty} f(x) = A \). Show that
\[
\lim_{t \to \infty} \frac{1}{t} \int_1^t f(x) \, dx = A
\]

(5) Define the function
\[
f(x) = \frac{2x}{1 + x^2}
\]
and let \( f_n(x) = f(nx) \).
(a) Show that the sequence \( f_n \) converges to 0 uniformly on the interval \([1, \infty)\).
(b) Does the sequence \( f_n \) converge to 0 on \([0, 1]\)? Is the convergence uniform?

(6) Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is differentiable, \( f(0) = 0 \), and \( f'(x) > f(x) \) for all \( x \in \mathbb{R} \). Prove that \( f(x) > 0 \) for \( x > 0 \).

Part 2. Complex Analysis. Answer 4 of the 5 questions in Part 2. If you answer more questions then indicate which you wish to be considered.

(1) (a) The symbol \( i^i \) is infinitely many valued. Find these values. They are all real positive.
(b) Solve the transcendental equation \( \cos z = \sqrt{3} \)

(2) Show that \( u(x, y) = 6x^2y + 6xy - 2y^2 \) is harmonic. Then find a function \( v(x, y) \) so that \( f(x + iy) = u(x, y) + iv(x, y) \) is analytic.

(3) Write the Laurent series in powers of \( z \) that represents the function
\[
f(z) = \frac{z - 1}{z(z^2 + 1)}
\]
in certain domains and specify those domains.

(4) (a) Evaluate the integral
\[
\int_0^{2\pi} \frac{d\theta}{5 - 3\sin \theta}
\]
(b) By integrating $\frac{e^{iz}}{z}$ over a suitable contour, evaluate
$$\int_{0}^{\infty} \frac{\sin x}{x} \, dx$$

(5) Evaluate the contour integrals. Here $C$ is the circle $\{z : |z - i| = 1\}$ oriented counterclockwise.

(a) $\int_{C} e^{z^2} + z^4 + \frac{z^3 + 4z}{2z - i} + \frac{1}{z^2 + 1} \, dz$

(b) $\int_{|z|=1} \frac{1}{z - \sin z} \, dz$