Please do four problems, including one from each of the three sections. Give complete proofs — do more than simply quote a theorem. Please indicate clearly which four problems you want to be graded.

Part I: Group theory

1. Let $G$ be an arbitrary group (not necessarily finite) and let $p > 0$ be a prime. Suppose $x \in G$ is an element of finite order $n = p^k m$ where $m$ is prime to $p$ (i.e., $m$ is not divisible by $p$).

   (a) Show that $x = yz = zy$ for some $y, z \in G$, where $y$ and $z$ both have finite order, the order of $y$ is a power of $p$, and the order of $z$ is prime to $p$.

   (b) Keeping the notation of part (a), suppose that $p = 2$, $k = 3$, and $m = 15$ (so $n = 2^3 15 = 120$). Find a pair of $y$ and $z$ as guaranteed in part (a), expressing them as powers of $x$.

2. Let $G$ be a finite group and assume that $H$ and $K$ are subgroups of $G$ such that the product of the orders of $H$ and $K$ is strictly greater than the order of $G$.

   (a) Prove that $H \cap K \neq \{1\}$ where $1 \in G$ is the identity element.

   (b) Now suppose that $K$ is a normal subgroup of $G$. What is the smallest possible order (in terms of the orders of $H$, $K$, and $G$) that $H \cap K$ could have?

Part II: Ring theory

3. An element $a$ of a ring is said to be nilpotent if $a^k = 0$ for some positive integer $k$.

   (a) Suppose that $n > 1$ is an integer and that every element of the ring $\mathbb{Z}/n\mathbb{Z}$ is either a unit or a nilpotent element. Prove that $n = p^m$ for some prime $p$ and positive integer $m$.

   (b) If $p$ is a prime and $m$ is a fixed positive integer, does the ring $\mathbb{Z}/p^m\mathbb{Z}$ have more units or more nilpotent elements? How many of each?

4. Let $g(x) = x^3 + 3x + 2 \in F[x]$, where $F = \mathbb{Z}/7\mathbb{Z}$ is the finite field with seven elements, and let $I = (g(x))$ be the ideal of $F[x]$ generated by $g(x)$. Let $K = F[x]/(g(x))$ and let $\alpha = x + I \in K = F[x]/(g(x))$.

   (a) Prove that $K$ is a field that contains a subfield isomorphic to $F$ and a root of $g(x)$.

   (b) Find a polynomial $p(x) \in F[x]$ such that $\alpha^{-1} = p(\alpha)$.

Part III: Linear algebra

5. (a) Let $V$ and $W$ be vector spaces and let $T$ be a linear operator from $V$ into $W$. Suppose that $V$ is finite-dimensional. Prove rank$(T) + $ nullity$(T) = \text{dim}(V)$.

   (b) Let $S$ be the linear operator defined on the space of $3 \times 3$ real matrices given by,

   $$ S(A) = A + A^t, $$

   where $A^t$ denotes the transpose of the matrix $A$. Determine the rank of $S$.

6. Let $A$ be the $4 \times 4$ real matrix

   $$
   
   \begin{pmatrix}
   0 & 4 & 2 & 1 \\
   1 & 0 & 1 & 1 \\
   0 & 0 & -2 & 0 \\
   0 & 0 & 0 & -2
   \end{pmatrix}
   $$

   (a) Find the characteristic polynomial of $A$ and the eigenvalues of $A$.

   (b) Find a basis for each eigenspace of $A$.

   (c) Find $J$, the Jordan canonical form of $A$.

   (d) Find an invertible $P$ such that $AP = PJ$. 