To get full credit you must show all your work.
This exam contains 6 real analysis and 6 complex variables questions.

Real Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. (a) Define the supremum of a bounded set $A \subset \mathbb{R}$.
   (b) Let $A$ and $B$ be bounded subsets of real numbers. Show that
   \[
   \sup(A \cup B) = \max\{\sup A, \sup B\}.
   \]

2. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $\int_a^b f(x)dx = 0$ for all $0 \leq a \leq b \leq 1$. Show that $f = 0$.

3. (a) Define uniformly continuous functions on $\mathbb{R}$.
   (b) Show that $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \sin(1/x)$ is not uniformly continuous.
   (c) Show that $g : (0, 1) \to \mathbb{R}$ defined by $g(x) = x \sin(1/x)$ is uniformly continuous on $(0, 1)$.

4. Prove the root test for series.
   (a) Show that if $\lim_{n \to \infty} |a_n|^{1/n} < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.
   (b) Give two examples of series $\sum_{n=1}^{\infty} c_n$ and $\sum_{n=1}^{\infty} d_n$ so that
   \[
   \lim_{n \to \infty} |c_n|^{1/n} = \lim_{n \to \infty} |d_n|^{1/n} = 1
   \]
   yet $\sum_{n=1}^{\infty} c_n$ is convergent while $\sum_{n=1}^{\infty} d_n$ is divergent.
5. Let \( f : [0, 1] \to [0, 1] \) be continuous. Show that there exists \( c \in [0, 1] \) such that \( f(c) = c \).

6. (a) Define compact subset of real numbers.
   (b) Let \( S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\} \). Using the definition of compactness to prove that \( S \) is a compact set.
   (c) Is \( \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \) compact? Prove your assertion.

**Complex Analysis**

100\% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. Find the image of line \( y = x \) under the mapping \( f(z) = z^2 - 1 \).
2. Let \( C \) denote the unit circle with counter-clockwise orientation. Compute the integral 
   \[ \oint_C \frac{e^{z^2}}{z^2} dz. \]
3. Expand \( f(z) = \frac{z^2}{(z+1)(z^2-1)} \) in a Laurent series valid for \( \{ z \in \mathbb{C} : 0 < |z-1| < 2 \} \).
4. Evaluate the integral \( \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + 1)^2(x^2 + 2x + 2)}. \)
5. Evaluate the line integral \( \int_C z^{1/2} dz \) where \( C \) is the positively oriented semicircular curve \( z = e^{i\theta} \) for \(-\pi/2 \leq \theta \leq \pi/2\) and \( z^{1/2} \) is defined with the standard branch cut \( |\text{Arg}(z)| < \pi \).
6. Let \( u \) and \( v \) be real valued functions continuously differentiable functions on a domain \( \Omega \). Assume that \( u \) is harmonic conjugate of \( v \) and \( v \) is harmonic conjugate of \( u \) on \( \Omega \). Show that both \( u \) and \( v \) are constant functions.