M.S. and M.A. Comprehensive Analysis Exam

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To get full credit you must show all your work.
This exam contains 6 real analysis and 6 complex variables questions.

Real Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. (a) Define the supremum of a bounded set \( A \subset \mathbb{R} \).
   (b) Let \( A = \left\{ \frac{1}{n} - \frac{1}{m} : n, m = 1, 2, 3, \ldots \right\} \). Find \( \text{sup} \ A \). Prove your claim.

2. (a) Define a uniformly continuous function on \( \mathbb{R} \).
   (b) Use the definition to show that \( f(x) = \frac{1}{x+1} \) is uniformly continuous on \([0, \infty)\).

3. (a) Let \( f \) be a bounded function on \([a, b]\). Define the Riemann integral \( \int_{a}^{b} f \).
   (b) Use the definition of Riemann integration to compute \( \int_{-1}^{1} f \) for \( f(x) = \begin{cases} 
0 & -1 \leq x \leq 0 \\
 x+1 & 0 < x \leq 1 
\end{cases} \).

4. Let \( x_n \geq 0 \) for all \( n \) and suppose that \( \lim_{n \to \infty} (-1)^n x_n \) exists. Show that \( \lim_{n \to \infty} x_n = 0 \).

5. Let \( (X, d_X), (Y, d_Y) \) be metric spaces and \( f : X \to Y \) be a continuous function. Show that if \( X \) is connected, then \( f(X) \) is connected.

6. Suppose that \( (X, d) \) is a complete metric space and \( \{ E_j : j \in \mathbb{N} \} \) is a collection of nonempty compact sets in \( X \) such that \( E_1 \supset E_2 \supset E_3 \supset \cdots \). Show that \( \bigcap_{j=1}^{\infty} E_j \neq \emptyset \).
Complex Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. Find the image of the square \( \{ x + iy : 0 \leq x \leq \ln 2, 0 \leq y \leq \pi \} \) under the mapping \( e^{2z} \).

2. Let \( C \) denote the unit circle with counter-clockwise orientation. Compute the integral \( \oint_C \text{Re}(z) \bar{z} \, dz \).

3. Expand \( f(z) = \frac{1}{z^2 - 1} \) in a Laurent series valid for \( \{ z \in \mathbb{C} : 0 < |z - 1| < 2 \} \).

4. Evaluate the integral \( \int_0^\infty \frac{1}{(1 + x^2)^2} \, dx \).

5. Find an entire function whose imaginary part is \( v(x, y) = x^2 - y^2 + 2 \).

6. (a) State Maximum Modulus Principle for analytic functions.

    (b) Let \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) be the unit disk and \( u : D \to \mathbb{R} \) be a harmonic function. Show that if \( u \) attains a maximum (or minimum) in \( D \) then it is constant. (Hint: use Maximum Modulus Principle for analytic functions.)