Do two problems from each of the three sections. Give complete proofs — do not just quote a theorem. Please indicate clearly which six problems you want graded.

Part I: Group theory

1. If $G$ is a group and $H$ is a subgroup of $G$ of index $n$, show that $G$ contains a normal subgroup $K$ whose index in $G$ divides $n!$.

2. (a) If $G$ is a group which contains only a finite number or subgroups, show that $G$ is finite.
   (b) Describe all groups $G$ which containing no proper subgroups.
   (c) Describe all groups $G$ which contain exactly one proper non-trivial subgroup.

3. Let $G$ be a finite $p$-group and let $H$ be a normal subgroup of $G$ of order $p$. Show that $H$ is contained in the center of $G$.

Part II: Ring theory

4. Let $\mathbb{Z}_3$ be the field with 3 elements. Find all monic irreducible polynomials of degree 3 in the ring $\mathbb{Z}_3[x]$.

5. Let $\mathbb{Z}$ be the ring of integers, $p$ a prime in $\mathbb{Z}$, and $\mathbb{Z}_p$, the field of $p$ elements. Let $x$ be an indeterminate, and set
   \[ R_1 = \mathbb{Z}_p[x]/(x^2 - 2), \quad R_2 = \mathbb{Z}_p[x]/(x^2 - 3). \]
   Determine whether the rings $R_1$ and $R_2$ are isomorphic in each of the following cases: $p = 2, 5, 11$.

6. Let $\mathbb{F}$ be field, and let $R$ be the subset of $2 \times 2$ matrices over $\mathbb{F}$ which commute with the matrix
   \[
   \begin{pmatrix}
   0 & 1 \\
   0 & 0
   \end{pmatrix}.
   
   (a) Prove the $R$ is a commutative ring.
   (b) Prove that $R \cong \mathbb{F}[x]/I$, where $I$ is the ideal of $\mathbb{F}[x]$ generated by $x^2$.

Part III: Linear algebra

7. Consider the $4 \times 4$ real matrix
   \[
   A = \begin{pmatrix}
   2 & 1 & -1 & 2 \\
   0 & 2 & 1 & 1 \\
   0 & 0 & 2 & 1 \\
   0 & 0 & 0 & 3
   \end{pmatrix}
   
   (a) Find $J$, the Jordan canonical form of $A$.
   (b) Find an invertible matrix $P$ so that $AP = PJ$.  

8. (a) Let $V$ and $W$ be vector spaces over a field $\mathbb{F}$, and let $T$ be a linear operator from $V$ into $W$. Suppose that $V$ is finite-dimensional. Prove $\text{rank}(T) + \text{nullity}(T) = \dim(V)$.

(b) Let $S$ be the linear operator defined on the space of $3 \times 3$ real matrices given by

$$S(A) = A - A^t,$$

where $A^t$ denotes the transpose of the matrix $A$. Determine $\text{rank}(S)$.

9. Let $A$ be a $3 \times 3$ matrix over the field $\mathbb{R}$ of real numbers and suppose that $\text{tr}(A) = 6$, $\text{tr}(A^2) = 14$ and $\text{det}(A) = 6$. Here $\text{tr}(A)$ and $\text{det}(A)$ denote the trace and determinant of $A$. Prove that $A$ is similar to the diagonal matrix

$$\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{pmatrix}.$$