Probability and Statistical Theory

MS Comprehensive Examination

April 14, 2018

Instructions:

Show all of your computations.
Prove all of your assertions or quote the appropriate theorems.
Books, notes, and calculators may be used.

You have three hours.
1. [25 pts] Suppose that $A_1$ and $A_2$ are independent uniform random variables on $[0, 1]$. Let $X = \max\{A_1, A_2\}$, $Y = \min\{A_1, A_2\}$ and $Z = A_1 + A_2$. Compute the following:

(a) The probability density function $f_X$

(b) The expectation $E(X)$.

(c) The conditional expectation $E(Y|X)$ (Hint: $P(Y \leq y, X \leq x) = P(X \leq x) - P(Y > y, X \leq x)$)

(d) The covariance Cov$(X, Y)$

(e) The probability density function $f_Z$.

(f) The expectation $E(Z)$ and variance $\text{Var}(Z)$.

(g) The covariance Cov$(A_1, Z)$ and correlation $\rho(A_1, Z)$
2. [25 pts] Let us have two independent random samples: \( X_1, \ldots, X_n \) is a sample from \( N(\mu_x, \sigma^2_x) \), and \( Y_1, \ldots, Y_m \) is a sample from \( N(\mu_y, \sigma^2_y) \).

(a) Write down a joint pdf for \( \{X_1, \ldots, X_n, Y_1, \ldots, Y_m\} \)

(b) Find a 4-dimensional sufficient statistic.

(c) Find the MLE of \( \sigma^2_x \) and \( \sigma^2_y \)

(d) Assume \( \sigma^2_x = \sigma^2_y = \sigma^2 \). Find the MLE for \( \sigma^2 \).

(e) Find a LR test statistic for testing \( H_0: \sigma^2_x = \sigma^2_y \)
3. (50 points, 5 points each) \( X_1, \cdots, X_n \overset{iid}{\sim} f(x|\theta) = \frac{3x^2}{\theta} \exp\left(-\frac{x^3}{\theta}\right) \) for \( x \geq 0 \) and \( \theta > 0 \).

a. Find the distribution of \( Y = \sum_{i=1}^{n} X_i^3 \).

b. Find \( E(Y) \) and \( \text{Var}(Y) \).

c. Find a complete and sufficient statistic for \( \theta \).

d. Find the maximum likelihood estimator \( \hat{\theta}_{\text{MLE}} \) for \( \theta \).

e. Explain whether \( \hat{\theta}_{\text{MLE}} \) is the UMVUE. If it is not, find the UMVUE \( \tilde{\theta} \).

f. Calculate the variance of the UMVUE \( \tilde{\theta} \).

g. Calculate the Cramér-Rao Lower Bound. Does the UMVUE \( \tilde{\theta} \) reach it?

h. Find a UMP level \( \alpha \) test \( \phi(T) \) for

\[ H_0 : \theta = \theta_0 \text{ vs } H_1 : \theta > \theta_0. \]

Write the rejection region \( R \) of this test using the test statistic \( T \) and the quantile of a well-known distribution.

Consider \( H_0 : \theta = \theta_0 \text{ vs } H_1 : \theta \neq \theta_0 \).

i. Explain why a UMP level \( \alpha \) test does not exist.

j. Find the likelihood ratio test statistic \( \Lambda \). Using the quantile of a well-known distribution, write the rejection region \( R \) of the likelihood ratio test so that its level is \( \alpha \).