Part I: Group theory

1. Let $G$ be a finite abelian group and let $z$ be the product of all of the elements in $G$.
   
   (a) Prove that $z^2 = 1$.
   
   (b) Give an example of $G 
eq 1$ where $z = 1$.
   
   (c) Give an example of $G$ where $z 
eq 1$.

2. In this problem $G$ is always a finite group.
   
   (a) Show that if the order of $G$ satisfies $|G| = p^n$ where $p$ is a prime integer and $n$ a positive integer, then the center $Z(G)$ is non-trivial.
   
   (b) Show that if the order of $G$ satisfies $|G| = pq$ where $p$ and $q$ are distinct primes, then $G$ cannot be a simple group (i.e., $G$ must contain a non-trivial normal subgroup).

Part II: Ring theory

3. Let $R$ be a commutative ring with an identity element. Under addition $R$ is, of course, an abelian group. Suppose that each subgroup of this group is, in fact, an ideal of $R$. Show that the ring $R$ is isomorphic to the ring of integers $\mathbb{Z}$, or to the integers modulo $n$, for some integer $n$.

4. Recall that an ideal $P$ in a commutative ring $R$ is prime if, whenever $x, y \in R$ are such that $xy \in P$, then either $x \in P$ or $y \in P$ (or possibly both).
   
   Let $\mathbb{Q}$ be the field of rational numbers, $\mathbb{Z}$ be the ring of integers, with $\mathbb{Q}[x]$ and $\mathbb{Z}[x]$ the corresponding polynomial rings.
   
   (a) Show that in $\mathbb{Q}[x]$ every prime ideal is a maximal ideal.
   
   (b) Exhibit a prime ideal in $\mathbb{Z}[x]$ that is not maximal.

Part III: Linear algebra

5. In this problem, $A$ is an $m \times m$ matrix over the field of real numbers such that $A^n = I$ (the identity matrix) for some positive integer $n$.
   
   (a) Prove that $A^2 = I$ if such an $A$ is symmetric.
   
   (b) Give an example of such an $A$ where $A^2 \neq I$. Of course, by part (a), this $A$ won’t be symmetric.

6. Let $A$ be the $4 \times 4$ real matrix
   
   \[
   \begin{pmatrix}
   0 & 4 & 0 & 4 \\
   1 & 0 & -1 & 2 \\
   0 & 0 & 0 & 4 \\
   0 & 0 & 1 & 0 \\
   \end{pmatrix}
   \]
   
   (a) Find the characteristic polynomial of $A$ and the eigenvalues of $A$.
   
   (b) Find a basis for each eigenspace of $A$.
   
   (c) Find $J$, the Jordan canonical form of $A$.
   
   (d) Find an invertible matrix $P$ such that $AP = PJ$. 