To get full credit you must show all your work.
This exam contains 6 real analysis and 6 complex variables questions.

Real Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. Define \( f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x^3 \cos \left( \frac{1}{x} \right) & \text{if } x \neq 0. \end{cases} \)
   Show that \( f' \) is continuous at 0 but \( f' \) is not differentiable at 0.

2. (a) Let \( f \) be a bounded function on \([a, b]\). Define the Riemann integral \( \int_a^b f \).
   (b) Define \( g(x) = \begin{cases} -1 & \text{if } x \text{ is irrational}, \\ x & \text{if } x \text{ is rational.} \end{cases} \)
   Show that \( g \) is not Riemann integrable on \([0, 1]\).

3. (a) Define what it means that a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is uniformly continuous on \( \mathbb{R} \).
   (b) Show that \( f(x) = \sin(x) \) is uniformly continuous on \( \mathbb{R} \).
   (c) Show that \( g(x) = \sin(x^2) \) is not uniformly continuous on \( \mathbb{R} \).

4. Let \((X, d)\) be a metric space and define \( \rho(x, y) = \frac{d(x, y)}{1 + d(x, y)} \) for all \( x, y \in X \). Show that \( \rho \) is a metric on \( X \).

5. For \( n = 1, 2, \ldots \) and \( 0 \leq x \leq 1 \), define \( f_n(x) = x^n \).
   (a) Fix \( 0 < a < 1 \). Show that the sequence \( \{f_n\} \) converges uniformly on \((0, a)\).
   (b) Is there a subsequence \( \{f_{n_k}\} \) that converges uniformly on \((0, 1)\)? Explain.

6. Let \((X, d)\) be a metric space and \( f : X \rightarrow \mathbb{R} \) be a continuous function. If \( X \) is compact, show that \( f \) is uniformly continuous.
Complex Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. Let \( f(z) = \frac{z - i}{z + i} \). Show that \( f \) maps the upper half-plane \( \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) into the unit disk \( \{ z \in \mathbb{C} : |z| < 1 \} \).

2. Evaluate the following integral, where \( C \) is the positively oriented circle centered at \((2, 0)\) with radius 2:
\[ \oint_C \frac{ze^{3z}}{(z^2 - 1)^2} \, dz. \]

3. Expand \( f(z) = \frac{z}{(z + i)(z - 3)} \) in a Laurent series valid for \( \{ z \in \mathbb{C} : 1 < |z| < 3 \} \).

4. Let \( f(z) = u(x, y) + iv(x, y) \) be an analytic function in the open unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \). Assume that \( u(x, y) \neq v(x, y) \) for all \( (x, y) \in D \). Find all such \( f \) for which the function
\[ g(z) = [u(x, y)]^2 + i[v(x, y)]^2 \]
is analytic in \( D \).

5. Evaluate the integral \( \int_0^\infty \frac{x^2}{(x^2 + 1)(x^2 + 9)} \, dx. \)

6. Find all possible entire functions \( f \) with the property that \( |f(z)| \leq 2|z| + 1 \) for all \( z \in \mathbb{C} \). Prove that you have found all such functions.