This exam has two parts: ordinary differential equations and partial differential equations. Choose four problems in each part.

**Part I: Ordinary Differential Equations**

1. Find the solution of the initial value problem on $(0, \infty)$. Draw the graph of the solution as a parametric curve:
   \[ \dot{y}(t) = y(t), \quad \dot{x}(t) = -5x(t) - 2y(t), \quad x(0) = 1, \quad y'(0) = -2 \]

2. Find three linearly independent solutions of the equation and show that these solutions are linearly independent.
   \[ y'''' + 3y''' + 3y'' + y = 0 \]

3. Prove that there are no continuous functions $p(t), q(t)$ on $\mathbb{R}$ so that $y_1(t) = e^t$ and $y_2(t) = t^2$ are both solutions for $y'' + p(t)y' + q(t)y = 0$.

4. Draw the phase portrait of the differential equation on the half plane $x \geq 0$. Find the smallest $v_0$ such that the solution $v(t)$ of the equation with the initial conditions $x(0) = 0, \dot{x}(0) = v_0$ satisfies $\lim_{t \to \infty} x(t) = \infty$

   \[ \dot{x} = \frac{24}{(6 + x)^2} \]

5. Let $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$
   1. Find the eigenvalues and corresponding eigenvectors of $A$.
   2. Draw phase portrait of $X' = AX$ near the origin and determine the stability of the origin.
   3. Solve $X' = AX$ by computing $e^{tA}$.

6. Compute eigenvalues and eigenfunctions of the following Sturm-Liouville problem.
   \[ y''(x) + \lambda y(x) = 0; \quad y(0) = y(2\pi), \quad y'(0) = y'(2\pi) \]

7. Suppose $y(t)$ has continuous first derivative and satisfies $|y'(t)| \leq 2y(t)$ on $[-1, 1]$ and $y(0) = 0$. Show that $y(t)$ is identically equal to zero on $[-1, 1]$. 
Part II: Partial Differential Equations

1. Find a solution of the initial value problem for the heat equation

\[
\begin{align*}
\left\{ \begin{array}{l}
u_t(x,t) = 4u_{xx}(x,t) \quad \text{on } -\infty < x < \infty, \ t > 0, \\
u(x,0) = 3e^{-2x^2} + 2 \quad \text{for } -\infty < x < \infty
\end{array} \right.
\]

2. Find the general solution for \( u_{tt} - 3u_{xt} + 2u_{tt} = 0 \) on \( \mathbb{R}^2 \)

3. Find all functions \( u(x, y) \) satisfying \( \Delta u + \lambda u = 0 \), where \( \lambda \) is a number, inside the unit square \([0, \pi] \times [0, \pi]\) and vanishing on its boundary

4. Solve the initial-boundary value problem for the wave equation

\[
\begin{align*}
\left\{ \begin{array}{l}
u_{tt}(x,t) = 9u_{xx}(x,t) \quad \text{on } 0 \leq x \leq \pi, \ t > 0, \\
u(0,t) = 0, \ u(\pi, t) = 0 \quad \text{for } t > 0, \\
u(x,0) = 3 \sin(x) - \tfrac{1}{4} \sin(3x) + 2 \sin(5x), \\
u_t(x,0) = 0 \quad \text{for } 0 \leq x \leq \pi
\end{array} \right.
\]

5. Show that the following boundary value problem has at most one solution

\[
\begin{align*}
\left\{ \begin{array}{l}
u_t = u_{xx}, \quad 0 < x < 1, \ t > 0 \\
u(x,0) = f(x), \quad 0 \leq x \leq 1 \\
u(0,t) = g(t), \quad t \geq 0 \\
u(1,t) = h(t), \quad t \geq 0
\end{array} \right.
\]

6. Let \( u(x, y) = (x, y) = -x^2 + y^2 + y^3 - 3x^2y \) Find \( v(x, y) \) satisfying the Cauchy-Riemann equations

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]

and the condition \( v(0,0) = 0 \)

7. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) Suppose a \( C^2 \) function \( u \) satisfies \( -\Delta u \leq 0 \) inside the domain and \( u \leq 0 \) on its boundary Prove that \( u(x) \leq 0 \) inside the domain