Please give complete proofs. Do two problems from each of the three parts. If you do three problems in one of the parts please indicate which two problems you want graded.

PART I.

1. Let $G$ be a finite abelian group and let $z$ be the product of all of the elements in $G$. Prove that $z^2 = 1$. Give an example of $G \neq 1$ where $z = 1$ and another example where $z \neq 1$.

2. Let $n \geq 3$ and let $H$ be the subgroup of the symmetric group $S_n$ which is generated by the set of 3-cycles. Show that $H$ is $A_n$, the alternating group on $n$ letters.

3. Let $G$ be a finite $p$-group and let $H$ be a normal subgroup of $G$ of order $p$. Show that $H$ is contained in the center of $G$.

PART II.

4. Prove that $\mathbb{Z}[x]$, the polynomial ring over the integers $\mathbb{Z}$, is not a principal ideal domain.

5. Let $R$ be a commutative ring with an identity element. Under addition $R$ is an abelian group. Suppose that each subgroup of this group is an ideal of $R$. Show that the ring $R$ is isomorphic to the ring of integers $\mathbb{Z}$, or to the integers modulo $n$, for some integer $n$.

6. Let $R$ be a commutative ring with an identity element and let $a$ be an element of $R$. Suppose that $M$ is an ideal of $R$ with the following two properties:

   (a) $a^n \notin M$ for $n = 1, 2, \cdots, .$

   (b) If $K$ is an ideal of $R$ which contains $M$ but $K \neq M$, then $a^n \in K$ for some $n$. Prove that $M$ is a prime ideal of $R$; that is, if $x, y \in R$ and $xy \in M$ then $x \in M$ or $y \in M$. 
PART III.

7. Let $A$ be a real symmetric $m \times m$ matrix and suppose that $A^n = I$ for some $n \geq 1$. Show that $A^2 = I$.

8. Let $T$ and $S$ be $n \times n$ matrices with entries from the field $K$. Suppose that $N$ is the null-space of $S$ (i.e., $N$ is the set of $n \times 1$ column vectors annihilated by $S$). Show that $TN \subseteq N$.

9. Let $T$ be an $n \times n$ matrix over the field $K$. Suppose that $p(x)$ is a nonzero polynomial of least degree with $p(T) = 0$. Show that $T$ is invertible if and only if the constant term of $p(x)$ is $\neq 0$. 