I. Ordinary Differential Equations

DO THREE of the following 5 problems.

1. (a) Solve the initial value problem for the differential equation.

\[
\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

for general \( \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \)

(b) Sketch a few sample trajectories.

2. Consider the initial value problem

\[
x'(t) = F(t, x), \quad x(t_0) = x_0
\]

where \(|t - t_0| < T\) for some \(T > 0\) and \(x_0 \in \mathbb{R}^n\). Suppose that \(F(t, x)\) is continuous in \((t, x)\) for all \(|t - t_0| < T\) and \(x \in \mathbb{R}^n\) and satisfies the Lipschitz condition

\[
|F(t, x) - F(t, y)| < L|x - y|
\]

for some \(L > 0\). Show that there exists a solution \(x(t)\) of the given initial value problem defined in a neighborhood of \(t_0\).

3. Again consider the initial value problem.

\[
x'(t) = F(t, x), \quad x(t_0) = x_0
\]

where \(F\) is as in the previous question, that is \(F(t, x)\) is continuous and is Lipschitz in \(x\). Show that, if any solution exists, then it must be unique.

4. Let \(y_1\) and \(y_2\) be two linearly independent solutions of the differential equation

\[
y'' + p(x)y' + q(x)y = 0
\]

where \(p(x)\) and \(q(x)\) are continuous real valued functions.

(a) Show that the zeroes of \(y_1\) are isolated (that is \(\{x : y_1(x) = 0\}\) has no accumulation points).

(b) Show that the Wronskian \(W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)\) is never zero.

(c) Show that, between any two successive zeroes of \(y_1\), there is exactly one zero of \(y_2\).

5. Find the solution of the initial value problem \(x' = Ax\) where

\[
A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}
\]

where \(x(0) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}\)

Remark: \(A\) is in “Jordan canonical form.”
II. Partial Differential Equations

DO THREE of the following 5 problems.

1. Solve $U = xU_x + yU_y + (U_x^2 + U_y^2)/3$ with initial condition $U(x, 0) = (1 - 2x^2)/2$.

2. Solve the initial value problem and verify your solution

   $$uu_x + yu_y = x, \quad u(x, 1) = 2x.$$

3. Suppose that $u(x)$ is a solution of

   $$\Delta u + au_x + bu_y = cu$$

   which is twice continuously differentiable on the open unit disk $D = \{(x, y) : x^2 + y^2 < 1\}$ and continuous on the closure $\overline{D}$ of $D$. Suppose that $a = a(x, y)$, $b = b(x, y)$ and $c(x, y)$ are continuous and $c(x, y) > 0$ on $\overline{D}$. Show that, if $u = 0$ on the boundary $\delta D$, then $u \equiv 0$ on $D$. [Suggestion: Show that $\max u \leq 0$ and $\min u \geq 0$.]

4. Solve the following initial value problem.

   $$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}_t + \begin{pmatrix} 2 & 8 \\ 8 & 2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   $U_1(x, 0) = \sin x$

   $U_2(x, 0) = \cos x$

5. Solve the initial/boundary value problem

   $$u_{tt} - u_{xx} = 1 \quad \text{for} \quad 0 < x < \pi \quad \text{and} \quad t > 0$$

   $u(x, 0) = 0, \quad u_t(x, 0) = 0 \quad \text{for} \quad 0 < x < \pi$

   $u(0, t) = 0, \quad u(\pi, t) = -\pi^2/2 \quad \text{for} \quad t \geq 0.$