Instructions: Do any six of the nine complete questions. No materials.

1. Suppose that \( f, g \in L^1(\mathbb{R}, m) \) where \( m \) denotes the Lebesgue measure and, for every \( a < b \)
\[
\int_a^b f \, dm \geq \int_a^b g \, dm.
\]
Show that \( m \)-almost everywhere, \( f \geq g \).

2. Suppose that \( g \in C([a, b]) \) and \( K \in C([a, b] \times [a, b]) \). For each \( u \in L^1([a, b], m) \) (\( m \) denotes Lebesgue measure on \([a, b]\)) define
\[
Tu(x) = g(x) + \int_a^b K(x, y) u(y) \, dm(y).
\]
(a) Show that \( Tu \in C([a, b]) \).
(b) Show that \( \{Tu : \|u\|_1 \leq 1\} \) is compact in \( C([a, b]) \). (Here \( \|\cdot\|_1 \) is the \( L^1([a, b], m) \) norm.

3. (a) State the Stone Weierstrass Theorem.
(b) Suppose that \( f \in C(0, \pi) \) and \( \int_0^\pi f(x) \cos nx \, dx = 0 \) for all \( n \).
Show that \( f(x) = 0 \) for all \( x, 0 < x < \pi \). (Suggestion: The trigonometric identity \( 2 \cos A \cos B = \cos(A + B) + \cos(A - B) \)
may be useful.)

4. Let \( f_n(x) = n(\sin x)^n \cos x \).
(a) Show that the sequence of functions \( f_n \) converges to 0 uniformly on any interval of the form \([0, a]\) where \( a < \pi/2 \).
(b) Does \( f_n \) converge to 0 uniformly on \([0, \pi/2]\)?
(c) Show that, for any continuous function \( g \in C([0, \pi/2]) \)
\[
\lim_{n \to \infty} \int_0^{\pi/2} f_n(x) g(x) \, dx = g(\pi/2).
\]
5. Prove or disprove each of the following three statements.

(a) If \( a_n \geq 0 \) is a sequence of nonnegative real numbers and \( \sum_{n=0}^{\infty} a_n \) exists then \( \sum_{n=0}^{\infty} a_n^2 \) exists.

(b) If \( a_n \) is a sequence of real numbers and \( \sum_{n=0}^{\infty} a_n \) exists then \( \sum_{n=0}^{\infty} a_n^2 \) exists.

(c) If \( a_n \) is a sequence of real numbers and \( \lim_{n \to \infty} a_n = 0 \) and the partial sums \( s_k = \sum_{n=1}^{k} a_n \) are uniformly bounded, then \( \sum_{n=1}^{\infty} a_n \) exists.

6. (a) Define the sets of first and second categories in topological space and state Baire category theorem.

(b) Show that the set of all transcendental numbers in the interval \([0,1]\) is a set of second category in \([0,1]\). A number that is a solution of a polynomial equation with integer coefficients is called an algebraic number, for example \( \sqrt{2} \). All others are called transcendental, for example \( e, \pi \) etc. are transcendental.

7. Define \( f(x) = \frac{1 - \cos x}{x} \) for \( x > 0 \).

(a) Show that \( f \notin L^1(0,\infty) \).

(b) Show that the improper Riemann integral \( \int_{1}^{\infty} \frac{\cos x}{x} \, dx \) exists.

8. Define the convergence of the infinite product \( \Pi_{n=1}^{\infty} (1 + a_n) \).

(a) Prove that \( \Pi_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right) \) diverges.

(b) \( \Pi_{n=1}^{\infty} \left( 1 + \frac{1}{n^2} \right) \) converges.
9. Prove or disprove each of the following statements.

(a) If $f_n \in L^1(\mu), f_n \geq 0$ for $n \in \mathbb{N}$ and $\{f_n\}$ converges pointwise to $f \in L^1(\mu)$ as $n \to \infty$, then

$$\lim_{n \to \infty} \int f_n d\mu = \int f d\mu.$$ 

(b) $f$ is measurable if and only if $|f|$ is measurable.

(c) Let $C$ be the middle-third Cantor set. Then the Lebesgue measure of $C$ is zero and the characteristic function of $C$ is Riemann integrable.