1. Suppose that $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Show that $f \in L^p(\mathbb{R})$ for all $p \geq 1$ and
\[
\lim_{p \to \infty} \|f\|_p = \|f\|_\infty
\]
where $\| \cdot \|_p$ denotes the $L^p(\mathbb{R})$ norm.

2. Let $(X, d)$ be a compact metric space. Suppose that $f : X \to X$ is an isometry (which means that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$). Show that $f$ is surjective.

3. (a) State the Arzela-Ascoli Theorem.
   
   (b) Suppose that $\{f_n : n \in \mathbb{N}\}$ is a sequence of functions in $C^1([0, 1])$ (so that $f_n$ is continuously differentiable on $[0, 1]$). Suppose further that, for all $n \in \mathbb{N}$
   \[
   |f'_n(x)| \leq \frac{1}{\sqrt{x}}, \text{ for all } x, 0 < x \leq 1
   \]
   and
   \[
   \int_0^1 f_n(x) \, dx = 0.
   \]
   Show that a subsequence of the sequence $f_n$ must converge uniformly on $[0,1]$.

4. Consider the sequence of functions $f_n(x) = \frac{ne^x}{1 + n^2 x^2}$, $n \in \mathbb{N}$, defined on $[0, 1]$. Evaluate the limit: $\lim_{n \to \infty} \int_0^1 f_n(x) \, dx$.

5. (a) Let $1 \leq p < \infty$. Show that, if a sequence of real-valued functions $\{f_n\}$, $n \geq 1$ converges in $L^p$, then a subsequence converges almost everywhere.
(b) Give an example of a sequence of functions converging to zero in $L^2(\mathbb{R})$ that does not converge almost everywhere.

6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $f(0) = 0$, and $f'(x) > f(x)$ for all $x \in \mathbb{R}$. Prove that $f(x) > 0$ for $x > 0$.

7. Let $p > 1$ and define $\phi : \mathbb{R} \to \mathbb{R}$, by

$$\phi(x) = \begin{cases} 
  x^{-1/p} & \text{if } 0 < x \leq 1 \\
  0 & \text{otherwise}
\end{cases}$$

Let $\{r_n : n \in \mathbb{N}\}$ be a countable dense subset of $\mathbb{R}$ and define

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} \phi(x - r_n)$$

(a) Prove that $f \in L^1(\mathbb{R})$ so that, in particular, $f(x) < \infty$ for almost all $x$.

(b) Prove that $f^p$ is not integrable on any interval $(a, b)$, $a < b$.

8. (a) State the Baire Category Theorem.
   (b) Suppose $\{f_n\}$ is a sequence of real valued continuous functions defined on a complete metric space $X$ and converging pointwise to 0. Show that given any $\epsilon > 0$, there exists a non-empty open set $U \subseteq X$ and a positive integer $N$ such that $|f_n(x)| < \epsilon$ for all $x$ in $U$ and all $n > N$. 