This exam has two parts, ordinary differential equations and partial differential equations. Choose four problems in each part.

**Part I: Ordinary Differential Equations**

1. Consider the sequence of functions \( \{y_k(t)\} \) defined by
   \[
   y_0(t) = 2 - 3t
   \]
   and
   \[
   y_{k+1}(t) = 2 - \int_0^t (3 + \cos(y_k(s)))\,ds, \quad k = 0, 1, \ldots.
   \]
   Prove that on any finite interval the sequence of the functions converges uniformly.

2. Let \( M \) be a manifold and \( \phi(t, \cdot) : M \to M \) be a family of diffeomorphisms with \( \phi(0, \cdot) \) equal to identity. (\( (M, \phi(t, \cdot)) \) is a dynamical system.) A set \( S \subset M \) is called **minimal** if 1) it is nonempty, closed, and invariant of the diffeomorphisms; and 2) it does not contain any such set as a proper subset. Prove the following result:
   If \( S \) is compact, then \( S \) is minimal if and only if for each \( x \in S \), we have \( S = \omega(x) \), where \( \omega(x) \) is the \( \omega \)-limit set of the curve \( \phi(t, x) \). (Recall that the \( \omega \)-limit set for \( \phi(t, x) \) is the set of all \( y \in M \) for which there exists an increasing sequence of times \( \{t_k\}_{k \in \mathbb{N}} \) with \( \lim_{k \to +\infty} t_k = +\infty \) such that \( \lim_{k \to +\infty} \phi(t_k, x) = y \).)

3. Consider the following initial value problem:
   \[
   \dot{x} = \begin{cases} 
   x \ln(|x|) & x \neq 0 \\
   0 & x = 0
   \end{cases}, \quad x(0) = x_0
   \]
   where \( x \in \mathbb{R} \).
   (a) Explain if there is a solution with \( x_0 = 0 \).
   (b) If a solution with \( x_0 = 0 \) exists, is it unique? Prove it or provide a counterexample.
4. Compute eigenvalues and eigenfunctions of the Sturm-Liouville problem. (Note that the eigenvalues are expressed in terms of the roots of a transcendental equation that cannot be solved exactly, so estimate graphically the position of these roots.)

\[ y''(x) + \lambda y(x) = 0, \ 0 < x < \pi, \]
\[ y'(0) - y(0) = 0, \ y(\pi) = 0. \]

5. Solve the nonhomogeneous linear system for \( x \in \mathbb{R}^2 \) with the initial condition.

\[ \dot{x} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} x + \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \]

6. Consider the nonlinear DE \( \ddot{x} + (x^2 - 1)\dot{x} + x = 0 \).
   (a) Determine all the equilibrium points in the phase plane.
   (b) Determine the stability of these equilibria by finding a suitable Lyapunov function.

7. Let \( \Sigma \) be the space of double sequences \( \{\sigma_k\}_{k \in \mathbb{Z}} \) with values in the alphabet \( \{0, 1\} \). Equip \( \Sigma \) with the metric

\[ \delta(\sigma, \sigma') = \sum_{k \in \mathbb{Z}} 2^{-|k|} \delta(\sigma_k, \sigma'_k), \quad \delta(\sigma_k, \sigma'_k) = \begin{cases} 0 & \sigma_k = \sigma'_k \\ 1 & \sigma_k \neq \sigma'_k \end{cases}. \]

Equip \( (\Sigma, \delta) \) with the dynamic \( \Psi : \Sigma \to \Sigma \) such that \( (\Psi(\sigma))_k = \sigma_{k+1} \), so that \( (\Sigma, \delta, \Psi) \) is a discrete dynamical system.
   (a) Prove that the periodic points for \( \Psi \) are dense in \( (\Sigma, \delta) \).
   (b) Prove that the dynamic is transitive, namely that there exist orbits that are dense in \( (\Sigma, \delta) \).
Part II: Partial Differential Equations

1. Find the solution to the heat equation with the initial-boundary values:

\[
\begin{aligned}
&u_t = u_{xx}, \quad \text{for } 0 < x < \pi, \ t > 0, \\
u(x, 0) = \frac{1}{3} \sin(\pi x) - 2 \sin(5\pi x), \quad \text{for } 0 < x < \pi, \\
u(0, t) = u(\pi, t) = 0, \quad \text{for } t > 0.
\end{aligned}
\]

2. Find all radially symmetric solutions of \(\Delta u - u = |x|^2\) in \(\mathbb{R}^3\).

3. Let us consider the following boundary value problem for the wave equation:

\[
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad (0 < x < l_1, \ 0 < t < l_2), \\
u(0, t) = 0, \ u(l_1, t) = 0 \quad (0 < t < l_2) \\
u(x, 0) = 0, \ u(x, l_2) = 0 \quad (0 < x < l_1).
\end{aligned}
\]

Assuming \(l_2/l_1\) is a rational number. Determine if the solution to the problem is unique or not in the given rectangle.

4. Let \(B^+\) be the open half disk \(\{(x, y) \mid x^2 + y^2 < 1, \ y > 0\}\). Suppose \(u(x, y) \in C^2(B^+) \cap C^1(\overline{B^+})\) satisfies \(\Delta u = u_{xx} + u_{yy} = 0\) in \(B^+\) and \(u_y(x, 0) = 0\). Prove that the extension \(u(x, y)\) to \(B\) defined as follows is a harmonic function on all \(B\).

\[
u(x, y) = \begin{cases} u(x, y) & \text{if } y \geq 0; \\
u(x, -y) & \text{if } y < 0.
\end{cases}
\]

5. Let \(Q\) be an open connected bounded domain and let \(\{u_n\}_{n \in \mathbb{N}}\) be a sequence of continuous functions on the closure \(\overline{Q}\). Suppose the functions are harmonic on \(Q\) and the sequence converges uniformly on \(\partial Q\).

(a) Prove that the sequence converges uniformly on all \(\overline{Q}\).

(b) Prove that the limit function is a \(C^2\) harmonic function on \(Q\).
6. Consider the following initial value problem

\[ \frac{\partial^2 u}{\partial t^2} = 4 \frac{\partial^2 u}{\partial x^2}, \quad (t > 0), \]
\[ u(x, 0) = f(x), \quad \frac{\partial u(x, 0)}{\partial t} = g(x), \quad (-\infty < x < \infty). \]

Suppose \( f(x) = g(x) = 0 \) for \( x \leq 0 \). Show that \( u(x, t) = 0 \) if \( x \leq -2t \) for \( t > 0 \).

7. Consider the initial-boundary value problem:

\[ u_t(x, t) + u_x(x, t) = x, \quad x > 0, \quad t > 0, \]
\[ u(0, t) = t, \quad t > 0, \]
\[ u(x, 0) = \sin(x), \quad x > 0. \]

Solve this problem using the method of characteristics. Is the solution continuous along the line \( x = t \)? What happens instead if we choose as initial value \( u(x, 0) = \cos(x) \)?