1. Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of real-valued Lebesgue measurable functions on \( \mathbb{R} \). Show that the following sets are measurable.

   (a) \( A = \{ x \in \mathbb{R} : \text{the sequence } \{f_n(x)\}_{n=1}^{\infty} \text{ is strictly increasing} \} \).

   (b) \( B = \{ x \in \mathbb{R} : \text{the sequence } \{f_n(x)\}_{n=1}^{\infty} \text{ is unbounded} \} \).

2. Let \( f_n(x) = nx^{n-1} - (n+1)x^n, x \in (0,1) \). Show that

\[
\int_{(0,1)} \sum_{n=1}^{\infty} f_n \, dm \neq \sum_{n=1}^{\infty} \int_{(0,1)} f_n \, dm
\]

and \( \sum_{n=1}^{\infty} \int_{(0,1)} |f_n| \, dm = \infty \).

3. Let \( A \subset \mathbb{R} \) be a set of finite Lebesgue measure. Suppose \( f : A \to [0, \infty) \) is integrable on \( A \). For \( \epsilon > 0 \), define

\[
S(\epsilon) = \sum_{k=0}^{\infty} k \epsilon \, m(A_k), \quad \text{where} \quad A_k = \{ x \in A : k \epsilon \leq f(x) < (k+1)\epsilon \}.
\]

Prove that

\[
\lim_{\epsilon \to 0} S(\epsilon) = \int_A f \, dm.
\]

4. Let \( S \) be the set of all functions \( f \) that are continuous on \([0,1]\) and differentiable on \((0,1)\) with \( \int_0^1 |f'|^2 \, dm \leq 1 \). Show that \( S \) has a compact closure in \( C([0,1]) \) with sup-norm \( \| \cdot \|_{\infty} \), where

\[
\|g\|_{\infty} = \sup_{0 \leq x \leq 1} |g(x)|.
\]
5. Let \( f \) be continuous on the interval \([0, 1]\). Show that

\[
\lim_{n \to \infty} n \int_{(0,1)} x^{2n} f(x) \, dm(x) = \frac{1}{2} f(1).
\]

Suggestion: consider first \( f \) a polynomial.

6. For a closed, bounded interval \([a, b]\), let \( \{f_n\} \) be a sequence in \( C([a, b]) \). If \( \{f_n\} \) is equicontinuous, does \( \{f_n\} \) necessarily have a uniformly convergent subsequence? If \( \{f_n\} \) is uniformly bounded, does \( \{f_n\} \) necessarily have a uniformly convergent subsequence?

7. Let \( f \) belong to \( L^p((0, \infty)) \) for some \( 1 \leq p < \infty \). Show that

\[
\lim_{x \to \infty} \frac{1}{x^2} \int_{(0,x)} tf(t) \, dm(t) = 0. \quad (*)
\]

Does (*) hold if \( f \) belongs to \( L^\infty((0, \infty)) \)? Explain.

8. Construct a sequence \( \{f_n\}_{n=1}^\infty \subset L^1([0, 1]) \) such that \( \|f_n\|_1 \to 0 \) as \( n \to \infty \) but for any \( t \in [0, 1] \), the sequence \( \{f_n(t)\}_{n=1}^\infty \) does not converge.