This exam has two parts, ordinary differential equations and partial differential equations. Choose four problems in each part.

Part I: Ordinary Differential Equations

1. Suppose that \( u(t), \alpha(t), \beta(t) \) are real-valued and continuous on \([a, b]\), \( \beta(t) \geq 0 \) on \([a, b]\), and

\[
u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds, \quad a \leq t \leq b.
\]

Prove:

\[
u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s)e^\int_s^t \beta(\tau)d\tau ds, \quad a \leq t \leq b.
\]

2. Consider the sequence of functions \( \{y_k(t)\}_{k=0}^\infty \) defined by

\[
\left\{
\begin{array}{ll}
y_0(t) &= 0, \\
y_{k+1}(t) &= 2 + \int_0^t \cos^2(\tau)y_k(\tau)d\tau, \quad k = 0, 1, \ldots
\end{array}
\right.
\]

Prove that \( \{y_k(t)\}_{k=0}^\infty \) converges uniformly on the finite interval \([-N, N]\) where \( N > 0 \).

3. Consider the following system in the plane minus the origin written in polar coordinates \((r, \theta)\):

\[
\dot{r} = 1 - r(1 + \cos(\theta)), \quad \dot{\theta} = 1.
\]

(a) Show that there exists an annulus \( D = \{(r, \theta) \in \mathbb{R}^2|0 < r_1 < r < r_2\} \) which is forward invariant for the system, determining suitable constants \( r_1 \) and \( r_2 \). This means that if a solution starts at time \( t_0 \) in \( D \), it will remain in \( D \) for all \( t \geq t_0 \).

(b) Conclude that there is a periodic orbit inside \( D \) invoking a suitable result from the theory of ODEs.
4. Let \( y_1(x) \) and \( y_2(x) \) be two linearly independent solutions of
\[
y'' + p(x)y' + q(x)y = 0,
\]
where \( p(x) \) and \( q(x) \) are continuous on \( \mathbb{R} \). Prove that all the roots of \( y_1(x) \) and \( y_2(x) \) are isolated, and \( y_1(x) \) has exactly one root between any two successive roots of \( y_2(x) \).

5. Consider the initial value problem
\[
\frac{dx}{dt} = \begin{cases} 
-x \ln |x|, & x \neq 0 \\
0, & x = 0
\end{cases}, \quad x(0) = x_0.
\]
where \( x \in \mathbb{R} \).
(a) Explain if there is a solution with \( x_0 = 0 \).
(b) If a solution with \( x_0 = 0 \) exists, is it unique? Prove it or provide a counterexample.

6. Solve the nonhomogeneous linear system for \( x \in \mathbb{R}^2 \) with the initial condition.
\[
\dot{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.
\]

7. Consider the linear system of differential equations \( dx/dt = Ax \). Suppose \( A \) is an \( n \times n \) upper-triangular matrix with all its diagonal elements equal to one. Show that any solution equals \( e^t \) multiplied by a polynomial of \( t \) of order less than \( n \).
Part II: Partial Differential Equations

1. Find a solution to the heat equation with the initial values:

\[ \begin{cases} u_t - 4u_{xx} = 0, & \text{for } -\infty < x < \infty, \ t > 0; \\ u(x,0) = 3e^{-x^2}, & \text{for } -\infty < x < \infty. \end{cases} \]

2. Let \( u \) be a positive harmonic function on the ball

\[ B_r(0) = \{ x \in \mathbb{R}^3 : |x| = \left( \sum_{i=1}^{3} x_i^2 \right)^{\frac{1}{2}} \leq r \}. \]

(a) Prove that for any \( |x| < r \),

\[ \frac{r(r - |x|)}{(r + |x|)^2} u(0) \leq u(x) \leq \frac{r(r + |x|)}{(r - |x|)^2} u(0). \]

(b) Show that if \( u \) is a positive harmonic function in \( \mathbb{R}^3 \), then \( u \) is a constant function.

3. Solve the following problem

\[ \begin{cases} u_{tt} - a^2 u_{xx} = 0, & x > 0, t > 0, \\ u(x,0) = g(x), u_t(x,0) = h(x), & x \geq 0, \\ u(0,t) = 0, & t > 0, \end{cases} \]

where \( a > 0 \).

4. Let \( \Omega \) be an open bounded domain in \( \mathbb{R}^n \), and \( T \) be a positive number. Let \( \Omega_T = \Omega \times (0,T] \). The parabolic boundary of \( \Omega_T \) is defined as \( \Gamma_T := \overline{\Omega} \setminus \Omega_T \). Assume that \( u(x,t) \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega_T}) \) satisfies

\[ u_t - \Delta u \leq 0 \text{ in } \Omega_T. \]

Prove that

\[ \max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u. \]

5. Find all radially symmetric solutions of \( \Delta u = n \) in \( \mathbb{R}^n \) for \( n \geq 3 \).
6. Let $\Omega = \mathbb{R}^n \times (0, \infty)$, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Suppose that $u$ solves $u_{tt} - a^2 \Delta u = 0$ in $\Omega$ where $a > 0$. Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$ and consider the cone $K = \{(x, t) : |x - x_0| \leq a(t_0 - t), 0 \leq t \leq t_0\}$. Prove that $u \equiv 0$ within $K$, if $u \equiv u_t \equiv 0$ on $\{(x, t) : |x - x_0| \leq at_0, t = 0\}$.

7. Consider the initial-boundary value problem

\[
\begin{align*}
&u_t(x, t) + u_x(x, t) = x, \quad x > 0, t > 0, \\
u(x, 0) = \sin x, \quad x > 0, \\
u(0, t) = t, \quad t \geq 0.
\end{align*}
\]

Solve this problem using the method of characteristics. Is the solution continuous along the line $x = t$? What happens instead if we choose as initial value $u(x, 0) = \cos x$?