September 29, 2018

Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
This is a closed book examination.
This is a three hour test.
1. Let \( X \geq 0 \) be a random variable on \((\Omega, \mathcal{A}, P)\) and \( \int_{\Omega} X \, dP = a, 0 < a < \infty \). Show the set function \( \nu \) defined on \( \mathcal{A} \) as follows.

\[
\nu(A) = \frac{1}{a} \int_{A} X \, dP
\]

is a probability measure on \( \mathcal{A} \).

2. Let \( \{X_n, n \geq 1\} \) be a sequence of random variables. Show \( X_n \overset{P}{\to} 0 \) if and only if

\[
\mathbb{E} \left( \frac{X_n^2}{1 + X_n^2} \right) \to 0
\]
3. [25 points] Let \( \{X_n : n \geq 1\} \) be a sequence of random variables and let \( c \) be a constant. Show that if the sequence \( \{X_n : n \geq 1\} \) converges in distribution to \( c \), then the sequence \( \{X_n : n \geq 1\} \) converges in probability to \( c \).

4. [25 points] Let \( \mathcal{P} = \{P_\theta, \theta \in \Theta\} \) be a family of distributions, where \( \theta \) is a \( p \)-dimensional parameter contained in a parameter space \( \Theta \). Suppose that the distributions \( P_\theta \) of \( \mathcal{P} \) have probability densities \( p_\theta = \frac{dP_\theta}{d\mu} \) with respect to a \( \sigma \)-finite measure \( \mu \). Let \( X_1, \ldots, X_n \) be a random sample drawn from a population with density \( p_\theta \). Write \( \mathbf{X} = (X_1, \ldots, X_n) \) and \( \mathbf{x} = (x_1, \ldots, x_n) \), where \( \mathbf{x} \) is the observed value of \( \mathbf{X} \). Show that a necessary and sufficient condition for a statistic \( \psi(\mathbf{X}) \) to be sufficient for \( \mathcal{P} \) is that for any fixed \( \theta \) and \( \theta_0 \), the ratio \( \frac{p_\theta(\mathbf{x})}{p_{\theta_0}(\mathbf{x})} \) is a function only of \( U(\mathbf{x}) \).