Real Analysis Qualifying Exam
Fall, 1997.
The exam has two parts.
Closed book, no notes.
Justify your answer with as much detail as possible.

Part A. (40 points total) Do any four (4) of the following six (6) problems.

1. Prove or disprove: there is a function $f : [0, 1] \to [0, 1]$ whose set of points of continuity consists of exactly the irrational numbers in $[0, 1]$.
2. Show that $\sum \dfrac{1}{1 + n^2 x}$ converges uniformly on any closed subinterval of $(0, 1]$, but does not converge uniformly on $[0, 1]$.
3. A mapping $f : \mathbb{R} \to \mathbb{R}$ is open if the image of each open set is open. Show that a continuous open mapping must be monotonic.
4. Let $\{E_n\}$ be a decreasing sequence of nonempty, closed subsets of a complete metric space X such that $\lim_{n\to\infty} \operatorname{diam} \quad E_n = 0$. Show that $\cap_1^\infty E_n$ consists of exactly one point.
5. Show that if $f(x) = \sin 1/x, x \neq 0$, and $f(0) = 0$, then f is Riemann integrable on $[-1, 1]$.
6. Suppose g is Lebesgue integrable on $[a, b]$ and $f(x) = \int_a^x g(t)dt, x \in [a, b]$. Show that f has bounded variation.

Part B. (80 points total) Do any four (4) of the following six (6) problems. (For any number selected, you must do all the lettered parts.)

1. Suppose $f : R \to R$ is differentiable. A point is a fixed point for f if $f(x) = x$.

   (a) Show that f has at most one fixed point if $|f'(x)| < 1$, for all $x \in R$.

   (b) Show that f has exactly one fixed point if $\sup_{x \in R} |f'(x)| < 1$.

2. Assume that f is twice continuously differentiable in the interval (a, ∞) and M_0, M_1, M_2 are the least upper bounds of $|f|, |f'|, |f''|$ respectively. Show that

$$M_1^2 \leq 4M_0M_2.$$

   What happens if the domain happens to be a finite interval?

3. A collection $\mathcal{F}$ of continuous functions defined on $[a, b]$ is equicontinuous if given $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon, \quad \text{if} \quad |x - y| < \delta, \quad \text{and} \quad f \in \mathcal{F}.$$

   Show that if $\mathcal{F}$ is equicontinuous and uniformly bounded on the closed, bounded interval $[a, b]$ then any sequence $\{f_n\} \subset \mathcal{F}$ has a uniformly convergent subsequence.

4. Suppose $\{f_n\}$ is a sequence of measurable functions such that $f_n(x) \to 0$, for almost all $x \in [0, 1]$. Fix $\epsilon > 0$ and define $N(x)$ to be the least positive integer such that $|f_n(x) < \epsilon$, for all $n \geq N(x)$. (If no such integer exists, define $N(x) = \infty$.) Show that $N(x)$ is measurable and there is an N such that $\mu(\{x : N(x) \geq N\}) < \epsilon$.

5. Let f be a bounded measurable function on a finite interval $[a, b]$. Let $\underline{R}(f)$ and $\overline{R}(f)$ denote the lower and upper Riemann integrals, respectively, of f, and let $\mathcal{L}(f)$ denote the Lebesgue integral of f. Show that

$$\underline{R}(f) \leq \mathcal{L}(f) \leq \overline{R}(f).$$

   Give an example to show that $\mathcal{L}(f)$ can be strictly between $\underline{R}(f)$ and $\overline{R}(f)$.

6. Using the Stone-Weierstrass theorem show that

$$\lim_{n \to \infty} \int_0^1 e^{inx} f(x) dx = 0,$$

   for any continuous function f on $[0, 1]$.
   Also show that the same result holds for any Lebesgue integrable function.