This exam has been checked carefully for errors. If you find what you believe to be an error in a question, report this to the proctor. If the proctor’s interpretation still seems unsatisfactory to you, you may alter the question so that in your view it is correctly stated, but not in such a way that it becomes trivial.

**SECTION 1**

Do 3 of the following 5 problems.

1. Prove that \((0, 1)\) is homeomorphic to \(\mathbb{R}\).

2. Let \(X\) be a topological space which is connected and locally path connected. Prove that \(X\) is path connected.

3. Prove that a connected space is path connected if and only if every path component is open.

4. Prove that a space \(Y\) is Hausdorff if and only if for every space \(X\) and every pair of continuous functions \(f : X \rightarrow Y\) and \(g : X \rightarrow Y\), the set \(\{x \in X | f(x) = g(x)\}\) is closed in \(X\).

5. Let \(f : X \rightarrow Y\) be a quotient map, with \(Y\) connected. Show that if \(f^{-1}(y)\) is connected for all \(y \in Y\), then \(X\) is connected.

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*Date: April 22, 2006.*
Do 3 of the following 5 problems.

1) Prove
   (a) Define \( f, g : X \to S^n \) are continuous and
       \( f(x) \neq -g(x) \) for all \( x \in X \), then \( f \) is homotopic to \( g \).
   (b) A continuous \( f : S^n \to S^n \) either has a fixed point or
       is homotopic to the antipodal map.

2) Let \( X \) be a connected, locally path-connected space.
   Suppose \( \pi_1(X) \) is finite. Show that every continuous
   map \( f : X \to S^1 \) is homotopic to a constant map.

3) Let \( X_1 \) and \( X_2 \) be two copies of \( S^2 \) and let \( N_1, S_1 \) and
   \( N_2, S_2 \) be the north and south poles of \( X_1 \) and \( X_2 \), re-
   spectively. Define \( X \) to be the quotient space obtained
   by identifying \( N_1 \) with \( N_2 \) and \( S_1 \) with \( S_2 \). Compute the
   fundamental group of \( X \) by using the Seifert-van Kam-
   pen theorem.

4) Let \( S^1 \) be the unit circle in \( \mathbb{R}^2 \) and \( p : X \to S^1 \) be a
   covering map with finitely many sheets. Prove that \( X \)
   is homeomorphic to \( S^1 \).

5) Let \( p : \tilde{X} \to X \) be the universal covering space of a space
   \( X \) and let \( f : X \to X \) be a continuous map.
      (a) Prove that there exist lifts of \( f \) to \( \tilde{X} \), that is, maps
          \( \tilde{f} : \tilde{X} \to \tilde{X} \) such that \( p \circ \tilde{f} = f \circ p \).
      (b) Suppose \( \tilde{f}_1, \tilde{f}_2 \) are lifts of \( f \) and there exists \( \tilde{x}_1, \tilde{x}_2 \)
          such that \( \tilde{f}_1(\tilde{x}_1) = \tilde{x}_1, \tilde{f}_2(\tilde{x}_2) = \tilde{x}_2 \) and \( p(\tilde{x}_1) = p(\tilde{x}_2) \).
          Prove that there exists a covering transformation
          \( \sigma : \tilde{X} \to \tilde{X} \) such that \( \tilde{f}_2 = \sigma \tilde{f}_1 \sigma^{-1} \).