Ph.D. Qualifying Examination
Probability and Statistical Theory

April 24, 2010

Instructions
Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
This is a closed book examination.
This is a three hour test.
1. Let \( f_n \) and \( g_n \) be integrable functions for a measure \( \mu \) with \( |f_n| \leq g_n \). Suppose that as \( n \to \infty \), \( f_n(x) \to f(x) \) and \( g_n(x) \to g(x) \) for almost all \( x \). If \( \int g_n d\mu \to \int g d\mu \), then \( \int f_n d\mu \to \int f d\mu \).

2. Let \( \mu = E(Y) \) denote the mean of a response variable \( Y \). When the response variable \( Y \) is subject to missing, we do not observe all \( Y_1, \ldots, Y_n \) in the sample. Let \( D \) represent the missing indicator variable which is equal to 1 if \( Y \) is observed and is equal to 0 if \( Y \) is missing, and let \( (Y_1, D_1), \ldots, (Y_n, D_n) \) denote a random sample of \( (Y, D) \). The complete-case sample mean of the observed \( Y \)-values is defined by
\[
\hat{\mu}_c = \frac{\sum_{i=1}^{n} D_i Y_i}{\sum_{i=1}^{n} D_i}.
\]
Let \( \pi = E(D) = P(D = 1) \) denote the probability of observing \( Y \) so that \( 1 - \pi \) represents the missing proportion of \( Y \). We assume \( \pi > 0 \) so that there is positive probability of observing \( Y \).

(a) Show that \( \hat{\mu}_c \) is a consistent estimator of \( \mu_1 = E(Y|D = 1) \).

(b) Find the asymptotic variance \( \sigma_c^2 \) of \( \hat{\mu}_c \).

(c) Find a consistent estimator of \( \sigma_c^2 \) in part (b).

(d) Show that \( \mu_1 \) is greater than \( \mu \) if \( \pi(y) = P(D = 1|Y = y) \) is a strictly increasing function in \( y \).

(e) Show that when missing is completely at random, \( \hat{\mu}_c \) is a consistent estimator of \( \mu \). In this case, find the asymptotic variance of \( \hat{\mu}_c \) and compare it with the variance of the full-data sample mean \( \bar{Y} = n^{-1} \sum_{i=1}^{n} Y_i \).
3. (a) For any two events $A, B \in \mathcal{F}$ prove that

$$2P(AB) \leq P(A) + P(B).$$

(b) For integrable $X$ prove that

$$E(X) = \int_0^\infty P(X > t) dt - \int_{-\infty}^0 P(X < t) dt.$$