1. Let $\phi \in L^\infty(\mathbb{R})$. (The measure on $\mathbb{R}$ is Lebesgue measure.) Show that
$$
\lim_{n \to \infty} \left( \int_{\mathbb{R}} \frac{|\phi(x)|^n}{1 + x^2} \, dx \right)^{1/n} = \|\phi\|_{\infty}
$$

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $\lim_{x \to \infty} f'(x) = A$ exists. Show that
$$
\lim_{x \to \infty} \frac{f(x)}{x} = A
$$

3. Let $K : [0, 1] \times [0, 1] \to \mathbb{R}$ be continuous. If $f \in L^1(0, 1)$, set
$$
(Tf)(x) = \int_0^1 K(x, y)f(y) \, dy,
$$
for all $x \in [0, 1]$. (a) Show $Tf \in C([0, 1])$. (b) Let $B$ be the unit ball of $L^1(0, 1)$ and show that $T(B)$ is relatively compact in $C([0, 1])$.

4. In $C[0, 1]$, let
$$
\mathcal{A} = \text{span}\{x^n(1 - x) : n \geq 1\}.
$$
Prove that $\mathcal{A}$ is an algebra whose uniform closure is $\{f \in C[0, 1] : f(0) = f(1) = 0\}$.

5. Suppose that $f \in L^1(\mathbb{R})$ and $A$ is a Borel subset of $\mathbb{R}$. Show that the mapping
$$
t \mapsto \int \chi_{A+t} f(x) \, dx
$$
is continuous from $\mathbb{R}$ to itself. Here $A+t = \{x+t : x \in A\}$. (Suggestion: Begin with the case that $A$ is an interval.)
6. Suppose that \( f_n, n \in \mathbb{N} \) is a sequence of functions defined on \( A \subseteq \mathbb{R} \) and uniformly convergent to a function \( f \). Assume \( x_0 \) is a limit point of \( A \) and \( \lim_{x \to x_0} f_n(x) \) exists for all \( n \in \mathbb{N} \). Prove that
\[
\lim_{n \to \infty} \lim_{x \to x_0} f_n(x) = \lim_{x \to x_0} f(x)
\]
(in the sense that one limit exists if and only if the other does and they are equal).

7. Let \( f \) be a non-negative element of \( L^1(0, \infty) \) and let \( A \) be a Borel subset of \( (0, \infty) \).

(a) Suppose the Lebesgue measure \( m(A) \) of \( A \) is finite. Prove that
\[
\lim_{n \to \infty} \int_A (f(x))^{1/n} dx = m(\{ x \in A : f(x) > 0 \})
\]

(b) Consider the case when \( m(A) = \infty \). What can be said about
\[
\lim_{n \to \infty} \int_A (f(x))^{1/n} dx
\]
in this case?

8. Suppose that \( f \in L^1(\mathbb{R}) \). Show that, for almost all \( x \),
\[
\lim_{h \to 0} \frac{1}{h} \int_{|y-x|<h} |f(y) - f(x)| dy = 0
\]
Suggestion: Begin with the case that \( f \) is continuous and of compact support.