This exam has two parts, ordinary differential equations and partial differential equations. Choose four problems in each part.

**Part I: Ordinary Differential Equations**

1. Consider the differential equation of the first order with initial condition

   \[
   \frac{dx}{dt} = F(t, x), \quad x(a) = x_0 \in \mathbb{R}^n
   \]

   where \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \) and \( F(t, x) = (F_1(t, x), F_2(t, x), \ldots, F_n(t, x))^T \). Suppose \( F(t, x) \) is continuous for \( a \leq t \leq b \) and \( x \in \mathbb{R}^n \) and satisfies a Lipschitz condition \( \|F(t, x) - F(t, y)\| \leq L\|x - y\| \) for \( a \leq t \leq b \) and all \( x, y \), where \( \|\cdot\| \) denotes the standard norm in \( \mathbb{R}^n \).

   (a) Convert the differential equation with the initial condition into an equivalent integral equation.

   (b) Set up the Picard iteration process and prove that the sequence converges uniformly on the interval \([a, b]\) to a limit function \( x_\infty(t) \).

   (c) Show that \( x_\infty(t) \) is a solution to the differential equation on \([a, b]\).

   (d) Establish that the solution to the differential equation with the given initial condition is unique.

2. A gradient system in \( \mathbb{R}^n \) is a system of ODEs of the form

   \[
   \frac{dx(t)}{dt} = -\nabla V(x(t))
   \]

   for some smooth function \( V(x) \) \( (x \in \mathbb{R}^n) \) and \( \nabla \) is the gradient operator. Recall that a periodic orbit for such a system is a non-constant solution \( x(t) \), such that \( x(t) = x(t + T) \) for some \( T > 0 \). Show that a gradient system can not have periodic orbits.
3. Find the power series solution of $y'' - xy = 0$ satisfying the initial conditions $y(0) = 2, y'(0) = 0$. Determine the radius of convergence for the power series solution.

4. Compute eigenvalues and eigenfunctions of the following Sturm-Liouville problem (the eigenvalues are expressed in terms of the roots of a transcendental equation that cannot be solved exactly, so estimate graphically the position of these roots):

$$y''(x) + \lambda y(x) = 0, \quad 0 < x < 1,$$

$$y'(0) - y(0) = 0, \quad y(1) = 0.$$  

5. Solve the nonhomogeneous linear system for $x \in \mathbb{R}^2$ with the initial condition.

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} t \\ 1 \end{bmatrix} x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$  

6. Consider the nonlinear DE $\ddot{x} + 9 \sin(x) = 0$.

(a) Find the integral curves and sketch the trajectories in the phase plane.

(b) Show that the solution satisfying the initial conditions $x(0) = \pi/6, \dot{x}(0) = 0$ is a periodic function of $t$ and find an expression for the period.

7. Find Green’s function $G(t, \tau)$ for the differential equation $u'' - 4u = 0$. (Recall that $G(t, \tau)$ satisfies 1) $G(t, \tau) = 0$ if $t < \tau$; 2) $G_{tt} + G = 0$ if $t > \tau$; and 3) $G(\tau, \tau) = 0, G_t(\tau, \tau) = 1$.)
Part II: Partial Differential Equations

1. Consider the Fourier series solution to the heat equation with an initial condition and a boundary condition

\[
\begin{align*}
\begin{cases}
    u_t &= u_{xx}, \quad 0 < x < \pi, \ t > 0, \\
u(x, 0) &= f(x), \\
u(0, t) &= \nu(\pi, t) = 0.
\end{cases}
\end{align*}
\]

(a) Derive the formal solution \( u(x, t) = \sum_{k=1}^{\infty} b_k e^{-k^2 t} \sin kx \) where the \( b_k \) are the coefficients of the Fourier series \( \sum_{k=1}^{\infty} b_k \sin kx \) for the continuous function \( f(x) \).

(b) Show that for every \( \delta > 0 \) this solution series converges uniformly in the region \( 0 \leq x \leq \pi, t \geq \delta \). Also show the same convergence for the corresponding series for \( u_t, u_x, u_{xx} \).

(c) Show that \( \lim_{t \to 0^+} u(x, t) = f(x) \) in the \( L^2 \) norm.

2. Find all radially symmetric solutions of \( \Delta u = 0 \) in \( \mathbb{R}^n (n \geq 2) \).

3. Find the equation of the surface \( z = z(x, y) \) which satisfies the first order PDE

\[
4yz(x, y) \frac{\partial z(x, y)}{\partial x} + \frac{\partial z(x, y)}{\partial y} + 2y = 0,
\]

and contains the ellipse \( y^2 + z^2 = 1, x + z = 2 \).

4. Let \( B^+ = \{(x, y) \mid x^2 + y^2 < 1, y > 0\} \) be the open half disk. Suppose \( u(x, y) \in C^2(B^+) \cap C^0(\bar{B}^+) \) satisfies \( \Delta u = u_{xx} + u_{yy} = 0 \) in \( B^+ \) and \( u(x, 0) = 0 \). Prove that the extension \( u(x, y) \) to \( B \) defined as follows is a harmonic function on all \( B \).

\[
u(x, y) = \begin{cases} u(x, y) & \text{if } y \geq 0; \\ -u(x, -y) & \text{if } y < 0. \end{cases}
\]

5. (Removable Singularity for Harmonic Functions) Suppose \( u(x) \) is a smooth harmonic function on a punctured neighborhood of the origin with the origin being a possible singularity. Prove that if \( |u(x)| \leq M \) for some constant \( M \) on the punctured neighborhood, then the singularity at the origin is removable.
6. Solve the following problem

\[ \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad (x > 0, \ t > 0), \]

\[ u(0, t) = 0, \quad u(x, 0) = f(x), \quad \frac{\partial u(x, t)}{\partial t} = g(x), \quad (x > 0). \]

7. Show that the following partial differential equation is elliptic inside the unit ball \( B_1 = \{ x \mid |x| < 1 \} \) and is hyperbolic outside the unit ball.

\[ \sum_{i,j=1}^{n} (\delta_{i,j} - x_i x_j) \frac{\partial^2}{\partial x_i \partial x_j} u = 0. \]