This exam has two parts, ordinary differential equations and partial differential equations. Choose four problems in each part.

**Part I: Ordinary Differential Equations**

1. Suppose that $u(t), \alpha(t)$ are real-valued and continuous on $[a, b]$, and

$$u(t) \leq C + \int_a^t [\alpha(s)u(s) + K] \, ds, \quad a \leq t \leq b,$$

where $\alpha(t) \geq 0$ on $[a, b]$, $C$ and $K$ are nonnegative real numbers.

Prove:

$$u(t) \leq [C + K(t - a)]e^{\int_a^t \alpha(s) \, ds}, \quad a \leq t \leq b.$$

2. Consider the sequence of functions $\{y_k(t)\}_{k=0}^\infty$ defined by

$$\begin{cases}
y_0(t) &= 2 - 3t, \\
y_{k+1}(t) &= 2 - \int_0^t (3 + \cos(y_k(\tau))) \, d\tau, \quad k = 0, 1, \ldots
\end{cases}$$

Prove that $\{y_k(t)\}_{k=0}^\infty$ converges uniformly on the finite interval $[-N, N]$ where $N > 0$.

3. Consider the planar system

$$\begin{cases}
\dot{x} &= f(x, y), \\
\dot{y} &= g(x, y).
\end{cases}$$

Suppose that: (H1) $f \in C^1(\Omega)$ and $g \in C^1(\Omega)$ where $\Omega \subset \mathbb{R}^2$ is simply connected; (H2) $f_x(x, y) + g_y(x, y)$ is positive almost everywhere in $\Omega$.

Prove: the system has no closed orbit lying entirely in $\Omega$.

4. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of

$$y'' + p(x)y' + q(x)y = 0,$$
where \( p(x) \) and \( q(x) \) are continuous on \( \mathbb{R} \). Prove that all the roots of \( y_1(x) \) and \( y_2(x) \) are simple and \( y_1(x) \) has exactly one root between any two successive roots of \( y_2(x) \).

5. Find the general solution of the differential equation

\[
y''' + 3y'' + 3y' - 7y = 0
\]
as a linear combination of three particular solutions \( y_1(t), y_2(t), y_3(t) \). Prove that the three particular solutions are linearly independent. That is, show that if \( c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) = 0 \) for some constants \( c_1, c_2, c_3 \) and for all \( t \), then \( c_1 = c_2 = c_3 = 0 \).

6. Solve the nonhomogeneous linear system for \( x \in \mathbb{R}^2 \) with the initial condition.

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} x + \begin{bmatrix} 2 - 2t \\ 1 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.
\]

7. Consider the linear system of differential equations \( dx/dt = Ax \). Suppose \( A \) is an \( n \times n \) upper-triangular matrix with all its diagonal elements equal to one. Show that any solution equals \( e^t \) multiplied by a polynomial of \( t \) of order less than \( n \).
Part II: Partial Differential Equations

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.
   (a) If $v \in C^2(\Omega) \cap C(\overline{\Omega})$ and $-\Delta v \leq 0$ in $\Omega$. Prove that $\max_{\overline{\Omega}} v = \max_{\partial \Omega} v$.
   (b) Prove that there exists a constant $C$ depending only on $\Omega$ such that
   \[ \max_{\overline{\Omega}} |u| \leq C(\max_{\partial \Omega} |g| + \max_{\overline{\Omega}} |f|) \]
   if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution of
   \[ \begin{cases} 
   -\Delta u = f & \text{in } \Omega, \\
   u = g & \text{on } \partial \Omega.
   \end{cases} \]
   (Hint: $-\Delta (u + \frac{|x|^2}{2n} \lambda) \leq 0$ for $\lambda = \max_{\overline{\Omega}} |f|$.)

2. Find a solution to the heat equation with the initial values:
   \[ \begin{cases} 
   u_t(x, t) - 9u_{xx}(x, t) = 0, & -\infty < x < \infty, \ t > 0; \\
v(x, 0) = 4e^{-2x^2}, & -\infty < x < \infty.
   \end{cases} \]

3. Let $a > 0$ and let $\Omega \subset \mathbb{R}^n$ be a connected, bounded open set with a smooth boundary $\partial \Omega$. If $u(x, t)$ is a $C^2$ function on $\overline{\Omega} \times [0, \infty)$, and solves the initial/boundary-value problem
   \[ \begin{cases} 
   u_{tt} - a^2 \Delta u = 0, & (x, t) \in \overline{\Omega} \times [0, \infty); \\
u(x, t) = 0, & (x, t) \in \partial \Omega \times (0, \infty); \\
u(x, 0) = 0, u_t(x, 0) = 0, & x \in \overline{\Omega}.
   \end{cases} \]
   Prove that $u$ is identically equal to zero on $\overline{\Omega} \times [0, \infty)$.

4. Solve the first-order PDE:
   \[ u_x + yu_y + zu_z = u, \quad u(0, y, z) = y^2 + z^2. \]

5. Let $u(x, y)$ be a positive harmonic function on the disk
   \[ B_r(0) = \{(x, y) : x^2 + y^2 \leq r^2\}. \]

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(a) Prove that for any $\rho = \sqrt{x^2 + y^2} < r$,

$$\frac{r - \rho}{r + \rho} u(0) \leq u(x, y) \leq \frac{r + \rho}{r - \rho} u(0).$$

(b) Show that if $u(x, y)$ is a positive harmonic function in $\mathbb{R}^2$, then $u$ is a constant function.

6. Consider the differential equation $u_{xx} + u_{yy} = 0$ with the boundary conditions $u(x, 0) = 0$, $u_y(x, 0) = g(x)$ in a neighborhood of the real line segment $\{(x, 0) \mid -1 < x < 1\}$.

(a) Show that there is a unique solution if $g(x)$ is a real analytic function. Name the theorem you use.

(b) Show that if a solution exists, then $g(x)$ must be a real analytic function.

7. Let $f(x)$ be the odd, periodic function with period $2\pi$ satisfying $f(x) = x$ on $0 \leq x \leq \pi/2$ and $f(x) = (\pi - x)$ on $\pi/2 \leq x \leq \pi$. Find the solution of the initial value problem for the wave equation.

$$\begin{cases}
    u_{tt}(x, t) - 4u_{xx}(x, t) = 0, & -\infty < x < \infty, \ t > 0; \\
    u(x, 0) = f(x), & -\infty < x < \infty; \\
    u_t(x, 0) = 0, & -\infty < x < \infty.
\end{cases}$$