Real Analysis Ph.D. Qualifying Exam Spring 95

Answer any six of the following questions to get full credit. Justify your answers with as much detail as possible.

1. Show that any complete metric space X is a set of second category in itself.

2. Let $\mathcal{A}$ be a closed subalgebra of the algebra $\mathcal{C}(\mathcal{T})$ of real-valued continuous functions on the compact space $\mathcal{T}$ under the sup norm.

   (a) Show that if $x \in \mathcal{A}$, then $|x|$ belongs to $\mathcal{A}$.

   (b) Assuming the truth of the Stone-Weierstrass theorem, show that the space generated by $\{1, \sin x, \sin 2x, \ldots \sin nx, \ldots\}$ is dense in $\mathcal{C}([0, \pi])$.

3. Consider a convex function $f : (a, b) \to \mathbb{R}$. Show that

   (a) f is Lipschitz on each closed subinterval of (a, b);

   (b) f has derivative everywhere except on a countable set.

4. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has the property that $x \to f(x, y)$ is in $\mathcal{L}(\mu)$,i.e., Lebesgue summable for each y in $\mathbb{R}$. Suppose $\frac{\partial y}{\partial x} f(x, y)$ exists and is finite for all (x, y) in $\mathbb{R}^2$ and that

$$|\frac{\partial f}{\partial x}(x, y)| \leq g(x)$$

   for each (x, y) and $g \in \mathcal{L}(\mu)$.

   (a) Show that

$$\frac{d}{dy}(\int_{-\infty}^{\infty} f(x, y)dx) = \int_{-\infty}^{\infty} \frac{\partial f}{\partial x}(x, y)dx$$

   where both integrals are Lebesgue.

   (b) Evaluate the integral

$$\int_0^{\infty} \frac{e^{-xt} - e^{-t}}{t} dt$$

   foe $x > 0$.

5. Assume that α and β are both positive. Let

$$f(x) = x^{\alpha} \sin(\frac{1}{x^{\beta}}) \quad \text{if} \quad 0 < x \leq 1 \quad \text{and} \quad 0 \quad \text{otherwise.}$$

   (a) Prove that f is of bounded variation on $[0, 1]$ if $\alpha > \beta$ bu not if $\alpha \leq \beta$.

   (b) Under what conditions on α, β would $f(x)$ be absolutely continuous.

6. Give a counterexample for those of the following statements which are not true.

   (a) Let $\mathcal{C}$ be a collection of closed sets of real numbers with the property that every finite subcollection of $\mathcal{C}$ has a non-empty intersection. Then
   $$\cap_{F\in\mathcal{C}} F \neq \emptyset.$$

   (b) Suppose f is continuous in $[-1,1]$. Then f is of bounded variation in $[-1,1]$.

   (c) Suppose f is continuous and monotone increasing on $[0.1]$. Then
   $$\int_0^1 f'(x)dx = f(1) - f(0).$$

   (d) Suppose the sequence $\{f_n\}$ converges in measure to f on $[0,1]$. Then $\{f_n\}$ converges at any x in $[0,1]$.

7. Give an example of a set E which is not Lebesgue measurable and also an example of a set F, which is Lebesgue measurable but not Borel measurable.

8. Determine all values of x for which the series
$$\sum_{n=1}^{\infty}(1 + 1/2 + 1/3 + \ldots + 1/n)\frac{\sin nx}{n}$$
converges.