INSTRUCTIONS: Do any four problems. And no more than four.

Please make sure that you give complete solutions to each problem that you do.

**Indicate which problems you wish to have graded.**

You have three hours.

**POLICY ON MISPRINTS**

The Ph.D. Qualifying Examination Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.
1. Let $G$ be a finite group, $p$ a prime, and $P$ a $p$-subgroup of $G$. Let $n$ be the number of Sylow $p$-subgroups of $G$ that contain $P$.

(a) [4 points] Prove $n \equiv 1 \mod p$.
(b) [4 points] Prove $n \geq |Syl_p(N_G(P))|$.
(c) [2 points] Prove that $G = S_4$ (the symmetric group on 4 letters) is an example of strict inequality in part (b).

2. Let $G$ be a finite simple group of order 168.

(a) [2 points] Show that $G$ has precisely 8 Sylow 7-subgroups.
(b) [3 points] Show that $G$ is isomorphic to a subgroup of $\tilde{G}$ of $A_8$ and that no element of order 2 in $\tilde{G}$ has a fixed point.
(c) [2 points] Show that $G$ has no element of order 6.
(d) [3 points] Find the number of Sylow 3-subgroups of $G$.

3. Find an orthogonal matrix $Q$ and a diagonal matrix $\Lambda$ so that $Q^{-1}AQ = \Lambda$ where

\[
\begin{pmatrix}
1 & -2 & 0 \\
-2 & 0 & 2 \\
0 & 2 & -1
\end{pmatrix}
\]

4. Let $R$ be a ring and $M$ be an $R$-module of finite composition length. If $f$ is an endomorphism of the $R$-module $M$, show that there is an integer $k$ such that $M = \Im f^k \oplus \Ker f^k$.

5. If $K = \mathbb{Q}(\sqrt{a})$, where $a$ is a negative integer and $\mathbb{Q}$ is the rationals, show that $K$ can’t be embedded in a cyclic extension whose degree over $\mathbb{Q}$ is divisible by 4.

6. Let $p$ be a prime and let $GF(p^m)$ denote a finite field of order $p^m$.

(a) [5 points] Show that $GF(p^m)$ is isomorphic to a subfield of $GF(p^n)$ if and only if $m$ divides $n$.
(b) [5 points] Let $E$ be the algebraic closure of $GF(p)$. Show that there is an intermediate field $L$ between $GF(p)$ and $E$ with $|L : GF(p)| = \infty$ and
$|E : L| = \infty$.

7. Let $R$ be a commutative ring with 1 and $R[x]$ the ring of polynomials in one (commuting) indeterminate with coefficients in $R$.

(a) [8 points] For each of the following statements indicate whether it is true or false. If it is false, give a counterexample. If it is true you do NOT have to provide a proof.

(i) If $R$ is a PID then so is $R[x]$.
(ii) If $R$ is a UFD then so is $R[x]$.
(iii) If $R$ is Artinian then so is $R[x]$.
(iv) If $R$ is Noetherian then so is $R[x]$.

(b) [2 points] What are the units in $R[x]$? Justify your answer.

8. Let $R$ be a ring with 1. If $M$ is an $R$-module, the uniform dimension of $M$ (ud $M$) is the largest integer $n$ such that there is a direct sum $M_1 \oplus \ldots \oplus M_n \subseteq M$ with all the $M_i$ non-zero. If no such integer exists then we say $\text{ud } M = \infty$. If $M \subseteq N$ are $R$-modules, $M$ is said to be essential in $N$ if every non-zero submodule of $N$ has non-zero intersection with $M$. Suppose the $\text{ud } M < \infty$ and $M \subseteq N$. Prove that $M$ is essential in $N$ if and only if $\text{ud } M = \text{ud } n$. 