No row swaps

Consider the system

$$\begin{aligned} 4x_1 + 2x_2 &= 10 \\ 6x_1 + 8x_2 &= 5 \end{aligned}$$

This has augmented matrix

$$\left(\begin{array}{cc|c} 4 & 2 & 10 \\ 6 & 8 & 5 \end{array} \right)$$

We proceed to put this into row echelon form using Gaussian Elimination:

$$\left(\begin{array}{cc|c} 4 & 2 & 10 \\ \mathbf{6} & 8 & 5 \end{array} \right) \;\; \rho_2 - (6/4)\rho_1 \;\; \left(\begin{array}{cc|c} 4 & 2 & 10 \\ 0 & 5 & -10 \end{array} \right)$$

No row swaps

Now we put this matrix into reduced row echelon form using Back Substitution:

$$\left(\begin{array}{cc|c} 4 & 2 & 10 \\ 0 & \mathbf{5} & -10 \end{array}\right) \quad \underset{(1/5)\rho_2}{} \quad \left(\begin{array}{cc|c} 4 & 2 & 10 \\ 0 & 1 & -2 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 4 & \mathbf{2} & 10 \\ 0 & 1 & -2 \end{array}\right) \quad \rho_1 - (2/1)\rho_2 \quad \left(\begin{array}{cc|c} 4 & 0 & 14 \\ 0 & 1 & -2 \end{array}\right)$$

$$\left(\begin{array}{cc|c} \mathbf{4} & 0 & 14 \\ 0 & 1 & -2 \end{array}\right) \quad (1/4)\rho_1 \quad \left(\begin{array}{cc|c} 1 & 0 & 7/2 \\ 0 & 1 & -2 \end{array}\right)$$

Now we can read off the solution:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7/2 \\ -2 \end{pmatrix}$$

No row swaps

Consider the system

$$\begin{array}{rcrcrcr} 3x_1 & - & 5x_2 & - & 2x_3 & = & 7 \\ 6x_1 & - & 8x_2 & - & x_3 & = & 5 \\ -9x_1 & + & 9x_2 & + & x_3 & = & -14 \end{array}$$

This has augmented matrix

$$\left(\begin{array}{rrr|r} 3 & -5 & -2 & 7 \\ 6 & -8 & -1 & 5 \\ -9 & 9 & 1 & -14 \end{array}\right)$$

We proceed to put this into row echelon form using Gaussian Elimination:

$$\left(\begin{array}{rrr|r} 3 & -5 & -2 & 7 \\ \mathbf{6} & -8 & -1 & 5 \\ -9 & 9 & 1 & -14 \end{array}\right) \quad \rho_2 - (6/3)\rho_1 \quad \left(\begin{array}{rrr|r} 3 & -5 & -2 & 7 \\ 0 & 2 & 3 & -9 \\ -9 & 9 & 1 & -14 \end{array}\right)$$

$$\left(\begin{array}{rrr|r} 3 & -5 & -2 & 7 \\ 0 & 2 & 3 & -9 \\ -\mathbf{9} & 9 & 1 & -14 \end{array}\right) \quad \rho_3 - (-9/3)\rho_1 \quad \left(\begin{array}{rrr|r} 3 & -5 & -2 & 7 \\ 0 & 2 & 3 & -9 \\ 0 & -6 & -5 & 7 \end{array}\right)$$

$$\left(\begin{array}{rrr|r} 3 & -5 & -2 & 7 \\ 0 & 2 & 3 & -9 \\ 0 & -\mathbf{6} & -5 & 7 \end{array}\right) \quad \rho_3 - (-6/2)\rho_{\mathbf{2}} \quad \left(\begin{array}{rrr|r} 3 & -5 & -2 & 7 \\ 0 & 2 & 3 & -9 \\ 0 & 0 & 4 & -20 \end{array}\right)$$

No row swaps

Now we put this matrix into reduced row echelon form using Back Substitution:

$$\left(\begin{array}{ccc|c} 3 & -5 & -2 & 7 \\ 0 & 2 & 3 & -9 \\ 0 & 0 & \mathbf{4} & -20 \end{array}\right) \quad (1/4)\rho_3 \quad \left(\begin{array}{ccc|c} 3 & -5 & -2 & 7 \\ 0 & 2 & 3 & -9 \\ 0 & 0 & 1 & -5 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 3 & -5 & -2 & 7 \\ 0 & 2 & \mathbf{3} & -9 \\ 0 & 0 & 1 & -5 \end{array}\right) \quad \rho_2 - (3/1)\rho_3 \quad \left(\begin{array}{ccc|c} 3 & -5 & -2 & 7 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 1 & -5 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 3 & -5 & \mathbf{-2} & 7 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 1 & -5 \end{array}\right) \quad \rho_1 - (-2/1)\rho_3 \quad \left(\begin{array}{ccc|c} 3 & -5 & 0 & -3 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 1 & -5 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 3 & -5 & 0 & -3 \\ 0 & \mathbf{2} & 0 & 6 \\ 0 & 0 & 1 & -5 \end{array}\right) \quad (1/2)\rho_2 \quad \left(\begin{array}{ccc|c} 3 & -5 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 3 & \mathbf{-5} & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{array}\right) \quad \rho_1 - (-5/1)\rho_2 \quad \left(\begin{array}{ccc|c} 3 & 0 & 0 & 12 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{array}\right)$$

No row swaps

$$\left(\begin{array}{ccc|c} \mathbf{3} & 0 & 0 & 12 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{array}\right) \xrightarrow{(1/3)\rho_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{array}\right)$$

Now we can read off the solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ -5 \end{pmatrix}$$

A row swap

Consider the system

$$\begin{array}{rcrcrcr} 3x_1 & - & 5x_2 & - & 2x_3 & = & 8 \\ 6x_1 & - & 10x_2 & - & x_3 & = & 7 \\ -9x_1 & + & 9x_2 & + & x_3 & = & -21 \end{array}$$

This has augmented matrix

$$\left(\begin{array}{rrr|r} 3 & -5 & -2 & 8 \\ 6 & -10 & -1 & 7 \\ -9 & 9 & 1 & -21 \end{array}\right)$$

We proceed to put this into row echelon form using Gaussian Elimination:

$$\left(\begin{array}{rrr|r} 3 & -5 & -2 & 8 \\ \mathbf{6} & -10 & -1 & 7 \\ -9 & 9 & 1 & -21 \end{array}\right) \quad \rho_2 - ({}^6\!/\!{}_3)\rho_1 \quad \left(\begin{array}{rrr|r} 3 & -5 & -2 & 8 \\ 0 & 0 & 3 & -9 \\ -9 & 9 & 1 & -21 \end{array}\right)$$

$$\left(\begin{array}{rrr|r} 3 & -5 & -2 & 8 \\ 0 & 0 & 3 & -9 \\ -\mathbf{9} & 9 & 1 & -21 \end{array}\right) \quad \rho_3 - ({}^{-9}\!/\!{}_3)\rho_1 \quad \left(\begin{array}{rrr|r} 3 & -5 & -2 & 8 \\ 0 & 0 & 3 & -9 \\ 0 & -6 & -5 & 3 \end{array}\right)$$

$$\left(\begin{array}{rrr|r} 3 & -5 & -2 & 8 \\ 0 & 0 & 3 & -9 \\ 0 & -\mathbf{6} & -5 & 3 \end{array}\right) \quad \begin{array}{c} \rho_3 \\ \rho_2 \end{array} \quad \left(\begin{array}{rrr|r} 3 & -5 & -2 & 8 \\ 0 & -6 & -5 & 3 \\ 0 & 0 & 3 & -9 \end{array}\right)$$

A row swap

Now we put this matrix into reduced row echelon form using Back Substitution:

$$\left(\begin{array}{ccc|c} 3 & -5 & -2 & 8 \\ 0 & -6 & -5 & 3 \\ 0 & 0 & \mathbf{3} & -9 \end{array}\right) \quad (1/3)\rho_3 \quad \left(\begin{array}{ccc|c} 3 & -5 & -2 & 8 \\ 0 & -6 & -5 & 3 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 3 & -5 & -2 & 8 \\ 0 & -6 & -\mathbf{5} & 3 \\ 0 & 0 & 1 & -3 \end{array}\right) \quad \rho_2 - (-5/1)\rho_3 \quad \left(\begin{array}{ccc|c} 3 & -5 & -2 & 8 \\ 0 & -6 & 0 & -12 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 3 & -5 & -\mathbf{2} & 8 \\ 0 & -6 & 0 & -12 \\ 0 & 0 & 1 & -3 \end{array}\right) \quad \rho_1 - (-2/1)\rho_3 \quad \left(\begin{array}{ccc|c} 3 & -5 & 0 & 2 \\ 0 & -6 & 0 & -12 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 3 & -5 & 0 & 2 \\ 0 & -\mathbf{6} & 0 & -12 \\ 0 & 0 & 1 & -3 \end{array}\right) \quad (1/-6)\rho_2 \quad \left(\begin{array}{ccc|c} 3 & -5 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 3 & -\mathbf{5} & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right) \quad \rho_1 - (-5/1)\rho_2 \quad \left(\begin{array}{ccc|c} 3 & 0 & 0 & 12 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

A row swap

$$\left(\begin{array}{ccc|c} \mathbf{3} & 0 & 0 & 12 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right) \xrightarrow{(1/3)\rho_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Now we can read off the solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$$

No Solution

Consider the system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 2 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 1 \\ 7x_1 & + & 8x_2 & + & 9x_3 & = & 4 \end{array}$$

This has augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 4 \end{array}\right)$$

We proceed to put this into row echelon form using Gaussian Elimination:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ \mathbf{4} & 5 & 6 & 1 \\ 7 & 8 & 9 & 4 \end{array}\right) \quad \rho_2 - (4/1)\rho_1 \quad \left(\begin{array}{rrr|r} 1 & 2 & 3 & 2 \\ 0 & -3 & -6 & -7 \\ 7 & 8 & 9 & 4 \end{array}\right)$$

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 2 \\ 0 & -3 & -6 & -7 \\ \mathbf{7} & 8 & 9 & 4 \end{array}\right) \quad \rho_3 - (7/1)\rho_1 \quad \left(\begin{array}{rrr|r} 1 & 2 & 3 & 2 \\ 0 & -3 & -6 & -7 \\ 0 & -6 & -12 & -10 \end{array}\right)$$

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 2 \\ 0 & -3 & -6 & -7 \\ 0 & -\mathbf{6} & -12 & -10 \end{array}\right) \quad \rho_3 - (-6/-3)\rho_2 \quad \left(\begin{array}{rrr|r} 1 & 2 & 3 & 2 \\ 0 & -3 & -6 & -7 \\ 0 & 0 & 0 & 4 \end{array}\right)$$

This last row corresponds to the equation $0 = 4$ that has no solution (obviously).

Infinite Number of Solutions

Consider the system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & -2 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 1 \\ 7x_1 & + & 8x_2 & + & 9x_3 & = & 4 \end{array}$$

This has augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 4 \end{array}\right)$$

We proceed to put this into row echelon form using Gaussian Elimination:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ \mathbf{4} & 5 & 6 & 1 \\ 7 & 8 & 9 & 4 \end{array}\right) \quad \rho_2 - (4/1)\rho_1 \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & -3 & -6 & 9 \\ 7 & 8 & 9 & 4 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & -3 & -6 & 9 \\ \mathbf{7} & 8 & 9 & 4 \end{array}\right) \quad \rho_3 - (7/1)\rho_1 \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & -3 & -6 & 9 \\ 0 & -6 & -12 & 18 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & -3 & -6 & 9 \\ 0 & \mathbf{-6} & -12 & 18 \end{array}\right) \quad \rho_3 - (-6/-3)\rho_2 \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & -3 & -6 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Infinite Number of Solutions

We use backsubstitution to put this matrix in reduced row echelon form.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & \mathbf{-3} & -6 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (1/-3)\rho_2 \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & \mathbf{2} & 3 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \rho_1 - (2/1)\rho_2 \quad \left(\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The nonzero rows of the our last matrix correspond to the system

$$\begin{array}{ccccccc} x_1 & & & - & x_3 & = & 4 \\ & & x_2 & + & 2x_3 & = & -3 \end{array}$$

The pivot columns of this last matrix are 1 and 2 so we see that x_1 and x_2 are *basic* variables and x_3 is a *free* variable.

In order to avoid doing anything silly, let's give the free variable a parameter name $x_3 = r$ and then write all the variables in terms of this parameter. We have:

$$\begin{aligned} x_1 &= 4 + r \\ x_2 &= -3 - 2r \\ x_3 &= r \end{aligned}$$

where r is arbitrary.

Efficiency and Accuracy

Efficiency

A system with n equations in n unknowns can be solved using Gaussian elimination and Back Substitution with approximately ${}^{2n^3}/_3$ *flops* (floating point operations). There are no known algorithms that do any better than this.

Accuracy

Unfortunately Gaussian Elimination is not accurate when executed on a computer using floating point arithmetic.

However a small modification (called *partial pivoting*) repairs this defect (for most systems). All one has to do is to swap rows during Gaussian Elimination so that the multipliers in the shear operations are never greater than 1 in absolute value.

There is no restriction on the multipliers in the shear operations during back substitution.

Partial Pivoting

Consider the system

$$\begin{array}{rcrcrcr} 6x_1 & - & x_2 & - & 14x_3 & = & -40 \\ -12x_1 & + & 10x_2 & + & 27x_3 & = & 27 \\ 18x_1 & - & 18x_2 & - & 12x_3 & = & -30 \end{array}$$

This has augmented matrix

$$\left(\begin{array}{rrr|r} 6 & -1 & -14 & -40 \\ -12 & 10 & 27 & 27 \\ 18 & -18 & -12 & -30 \end{array}\right)$$

We proceed to put this into row echelon form using Gaussian Elimination with Partial Pivoting:

$$\left(\begin{array}{rrr|r} \mathbf{6} & -1 & -14 & -40 \\ -12 & 10 & 27 & 27 \\ \mathbf{18} & -18 & -12 & -30 \end{array}\right) \begin{array}{c} \rho_3 \\ \\ \rho_1 \end{array} \left(\begin{array}{rrr|r} 18 & -18 & -12 & -30 \\ -12 & 10 & 27 & 27 \\ 6 & -1 & -14 & -40 \end{array}\right)$$

$$\left(\begin{array}{rrr|r} 18 & -18 & -12 & -30 \\ \mathbf{-12} & 10 & 27 & 27 \\ 6 & -1 & -14 & -40 \end{array}\right) \; \rho_2 - (-12/18)\rho_1 \; \left(\begin{array}{rrr|r} 18 & -18 & -12 & -30 \\ 0 & -2 & 19 & 7 \\ 6 & -1 & -14 & -40 \end{array}\right)$$

Partial Pivoting

$$\left(\begin{array}{rrr|r} 18 & -18 & -12 & -30 \\ 0 & -2 & 19 & 7 \\ \mathbf{6} & -1 & -14 & -40 \end{array}\right) \quad \rho_3 - (6/18)\rho_1 \quad \left(\begin{array}{rrr|r} 18 & -18 & -12 & -30 \\ 0 & -2 & 19 & 7 \\ 0 & 5 & -10 & -30 \end{array}\right)$$

$$\left(\begin{array}{rrr|r} 18 & -18 & -12 & -30 \\ 0 & \mathbf{-2} & 19 & 7 \\ 0 & \mathbf{5} & -10 & -30 \end{array}\right) \quad \begin{array}{c} \rho_3 \\ \rho_2 \end{array} \quad \left(\begin{array}{rrr|r} 18 & -18 & -12 & -30 \\ 0 & 5 & -10 & -30 \\ 0 & -2 & 19 & 7 \end{array}\right)$$

$$\left(\begin{array}{rrr|r} 18 & -18 & -12 & -30 \\ 0 & 5 & -10 & -30 \\ 0 & \mathbf{-2} & 19 & 7 \end{array}\right) \quad \rho_3 - (-2/5)\rho_2 \quad \left(\begin{array}{rrr|r} 18 & -18 & -12 & -30 \\ 0 & 5 & -10 & -30 \\ 0 & 0 & 15 & -5 \end{array}\right)$$

After applying back substitution to this matrix we obtain the solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -77/9 \\ -20/3 \\ -1/3 \end{pmatrix}$$

(I think.)