Exam 1
Math 1890-001
Spring 2012 **Name** ______________________________

Instructions: No books. No notes. Non-graphing calculators only. Please write neatly. There are 12 problems on 10 pages. **Show your work! Explain your answers.**

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1. Let $\alpha = 4$, $\beta = 8$, $\vec{v} = \begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} -1 \\ -7 \\ 6 \end{pmatrix}$.

   Compute the linear combination $\alpha\,\vec{v}+\beta\,\vec{w}$, or explain why it is impossible. Show your work.

   answer: The linear combination $\alpha\,\vec{v}+\beta\,\vec{w} = \begin{pmatrix} -24 \\ -36 \\ 44 \end{pmatrix}$.

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2. Find the angle between the following two vectors, if it is defined. Show your work.

$$v = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix},$$

   answer: We will need the following information:

$$\begin{aligned} v \cdot w &= (1)(0) + (2)(4) + (0)(1) = 8 \\ \|v\| &= \sqrt{(1)(1) + (2)(2) + (0)(0)} = \sqrt{5} \\ \|w\| &= \sqrt{(0)(0) + (4)(4) + (1)(1)} = \sqrt{17} \end{aligned}$$

   The answer is the angle

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{v \cdot w}{\|v\|\,\|w\|}\right) \\ &= \cos^{-1}\left(\frac{8}{\sqrt{5}\,\sqrt{17}}\right) \\ &= \cos^{-1}\left(\frac{8}{\sqrt{85}}\right) \\ &\approx 0.52019588578126996 \text{ radians} \\ &\approx 29.805028775336197 \text{ degrees} \end{aligned}$$

3. Let $A = \begin{pmatrix} -3 & -3 & 7 \\ 9 & 3 & -6 \\ 9 & -1 & 8 \end{pmatrix}$ and $b = \begin{pmatrix} 19 \\ -54 \\ -48 \end{pmatrix}$.

Solve the linear system that has the augmented matrix $(A \,|\, b)$. Show your work.

answer: Simply row reduce the augmented matrix.

$$\left(\begin{array}{rrr|r} -3 & -3 & 7 & 19 \\ 9 & 3 & -6 & -54 \\ 9 & -1 & 8 & -48 \end{array}\right) \sim \left(\begin{array}{rrr|r} -3 & -3 & 7 & 19 \\ 0 & -6 & 15 & 3 \\ 0 & -10 & 29 & 9 \end{array}\right)$$

$$\sim \left(\begin{array}{rrr|r} -3 & -3 & 7 & 19 \\ 0 & -6 & 15 & 3 \\ 0 & 0 & 4 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{rrr|r} -3 & -3 & 0 & 12 \\ 0 & -6 & 0 & -12 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{rrr|r} -3 & 0 & 0 & 18 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{rrr|r} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

The solution is $x = \begin{pmatrix} -6 \\ 2 \\ 1 \end{pmatrix}$.

4. Let $A = \begin{pmatrix} -1 & -4 & 3 & -4 \\ 2 & 8 & 1 & 0 \\ 5 & 20 & 4 & 4 \end{pmatrix}$ and $b = \begin{pmatrix} -16 \\ -7 \\ -3 \end{pmatrix}$.

Find the general solution of the equation that has the augmented matrix $(A \,|\, b)$. Show your work.

HINT: The augmented matrix $(A \,|\, b)$ has reduced echelon form

$$\left(\begin{array}{cccc|c} 1 & 4 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

.

answer: There is only one free variable: x_2. The equations corresponding to the nonzero rows of the echelon form matrix are

$$\begin{aligned} x_1 &= -3 - 4x_2 \\ x_3 &= -1 \\ x_4 &= 4 \end{aligned}$$

The general solution is $x = \left\{ \begin{pmatrix} -3 \\ 0 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 \quad \text{where } x_2 \in \mathbb{R} \right\}$.

5. Let $A = \begin{pmatrix} -4 & 0 & 6 \\ 7 & 5 & 5 \\ -4 & -1 & -4 \\ 8 & -2 & 1 \end{pmatrix}$. Find A^T, the transpose of A.

answer: The transpose $A^T = \begin{pmatrix} -4 & 7 & -4 & 8 \\ 0 & 5 & -1 & -2 \\ 6 & 5 & -4 & 1 \end{pmatrix}$.

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6. Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. Find the rank of the matrix A. Explain your answer.

answer: Since $\text{Rowspace}(A) = \{(0,0)\} = \{\vec{0}\}$ (and this has basis $\emptyset$) the rank of A is 0.

7. For each part of this problem determine whether the given subset of $\mathbb{R}^3$ is linearly independent. Explain your answers.

(a) $S_1 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

(b) $S_2 = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix} \right\}$

(c) $S_3 = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right\}$

(d) $S_4 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \right\}$

answer: We'll use the condition that a set S contained in a vector space V is linearly independent if the $\vec{0}$ vector of V can only be represented as a linear combination of the vectors of S using coefficients that are all 0.

(a) No. No set containing the zero vector is ever linearly independent. Here, for example, $5 \cdot \vec{0} = \vec{0}$.

(b) No. Since the second vector is a scalar multiple of the first vector. In particular $3 \cdot \vec{v_1} + 1 \cdot v_2 = \vec{0}$.

(c) Yes. Since the homogeneous equation $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{0}$ has only the trivial solution $c_1 = c_2 = c_3 = 0$. To see this row reduce the matrix with columns consisting of the vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$. Since the echelon form has a leading variable in every column we have linear independence.

(d) No. Since the dimension of $\mathbb{R}^3$ is 3 there can't be a linearly independent subset with more than 3 elements.

8. For each part of this problem determine whether the given subset of $\mathbb{R}^3$ is a spanning set for $\mathbb{R}^3$. Explain your answers. *In case you didn't notice, these are the same sets of vectors from the previous problem.*

(a) $S_1 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

(b) $S_2 = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix} \right\}$

(c) $S_3 = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right\}$

(d) $S_4 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \right\}$

answer: We'll use the fact that a set S contained in a vector space V can't possibly be a spanning set for V if it contains fewer than $\dim(V)$ vectors. In this problem $\dim(\mathbb{R}^3) = 3$ so only sets with at least 3 vectors have a chance of spanning $\mathbb{R}^3$.

(a) No. Too few vectors

(b) No. Too few vectors

(c) Yes. From the previous problem we know that these vectors are linearly independent and so span a 3-dimensional subspace of $\mathbb{R}^3$. Clearly this subspace must be all of $\mathbb{R}^3$.

(d) Yes. The matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -5 \end{pmatrix}$ and so has rank 3. Consequently the columns span a 3-dimensional subspace of $\mathbb{R}^3$. Clearly this subspace must be all of $\mathbb{R}^3$.

9. For each part of this problem determine whether the given subset of $\mathbb{R}^3$ is a basis for $\mathbb{R}^3$. Explain your answers. *In case you didn't notice, these are the same sets of vectors from the previous two problems.*

(a) $S_1 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

(b) $S_2 = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix} \right\}$

(c) $S_3 = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right\}$

(d) $S_4 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \right\}$

answer: We'll use the fact that a basis S for a vector space V must be a linearly independent set with $\dim(V)$ vectors. In this problem $\dim(\mathbb{R}^3) = 3$ so only linearly independent sets with exactly 3 vectors are bases for $\mathbb{R}^3$.

(a) No. Wrong number of vectors.

(b) No. Wrong number of vectors.

(c) Yes. From problem 7 we know that these 3 vectors are linearly independent and so are a basis for $\mathbb{R}^3$.

(d) No. Wrong number of vectors.

10. List all the reduced echelon forms possible for a 3×3 matrix. Explain your answer.

answer: There are 8 families of these matrices. Each family is determined by which columns have leading 1s. Four of the families have a single member, the rest have and infinite number of members. In the list given below a, b are arbitrary numbers.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ no leading 1s

$$\begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ a single leading 1; in the first column

$$\begin{pmatrix} 0 & 1 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ a single leading 1; in the second column

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ a single leading 1; in the third column

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{pmatrix}$$ two leading 1s; in the first and second columns

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ two leading 1s; in the first and third columns

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ two leading 1s; in the second and third columns

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ three leading 1s; one in each column

11. Let $A = \begin{pmatrix} -26220 & -26220 & -4956 & -124704 & 6396 & 1257 \\ 8099 & 8099 & 1531 & 38520 & -1975 & -388 \\ 2486 & 2486 & 470 & 11824 & -606 & -119 \\ -714 & -714 & -135 & -3396 & 174 & 34 \\ -270 & -270 & -51 & -1284 & 66 & 13 \end{pmatrix}$.

Find bases for Rowspace(A) and Colspace(A). What is the rank of A? Explain your answers.

HINT: The reduced echelon form of A is

$$R = \begin{pmatrix} 1 & 1 & 0 & 4 & -1 & 0 \\ 0 & 0 & 1 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

answer: To get a basis for the row space of the matrix we take the nonzero rows of the echelon form of the matrix. So the rows of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 4 & -1 & 0 \\ 0 & 0 & 1 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

form a basis for Rowspace(A).

To get a basis for the column space of the matrix we take the columns that correspond to columns of the echelon form matrix that have leading ones. These are columns 1, 3, and 6, so the columns of the matrix

$$\begin{pmatrix} -26220 & -4956 & 1257 \\ 8099 & 1531 & -388 \\ 2486 & 470 & -119 \\ -714 & -135 & 34 \\ -270 & -51 & 13 \end{pmatrix}$$

form a basis for Colspace(A).

12. Suppose that V and W are subspaces of $\mathbb{R}^9$. Further suppose that

$$\dim(V) = 6, \ \dim(W) = 5, \text{ and } \dim(V \cap W) = 3.$$

Does $V + W = \mathbb{R}^9$? Explain your answer.

answer: No.

We know that

$$\begin{aligned} \dim(V + W) &= \dim(V) + \dim(W) - \dim(V \cap W) \\ &= 6 + 5 - 3 \\ &= 8 \end{aligned}$$

But the dimension $\mathbb{R}^9$ is $9 > 8$, so $V + W$ must be a *proper* subset of $\mathbb{R}^9$, i.e., $V + W \neq \mathbb{R}^9$.