Exam 2
Math 1890-001
Spring 2012 **Name** ______________________

Instructions: No books. No notes. Non-graphing calculators only. Please write neatly. There are 8 problems on 6 pages.
Show your work!

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1. (10 points) Let f be an automorphism of $\mathbb{R}^2$ with

$$f(\begin{pmatrix}1\\3\end{pmatrix}) = \begin{pmatrix}2\\-1\end{pmatrix} \quad \text{and} \quad f(\begin{pmatrix}1\\4\end{pmatrix}) = \begin{pmatrix}0\\1\end{pmatrix}.$$

Find

$$f(\begin{pmatrix}0\\-1\end{pmatrix}).$$

answer:

$$\begin{aligned} f(\begin{pmatrix}0\\-1\end{pmatrix}) &= f(\begin{pmatrix}1\\3\end{pmatrix} - \begin{pmatrix}1\\4\end{pmatrix}) \\ &= f(\begin{pmatrix}1\\3\end{pmatrix}) - f(\begin{pmatrix}1\\4\end{pmatrix}) \\ &= \begin{pmatrix}2\\-1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} \\ &= \begin{pmatrix}2\\-2\end{pmatrix}. \end{aligned}$$

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2. (15 points) Give the size of the product or state "not defined".

(a) a 2×3 matrix times a 3×1 matrix

answer: 2×1

(b) a 1×12 matrix times a 12×1 matrix

answer: 1×1

(c) a 2×3 matrix times a 2×1 matrix

answer: not defined

(d) a 2×2 matrix times a 2×2 matrix

answer: 2×2

(e) a 3×1 matrix times a 1×4 matrix

answer: 3×4

3. (20 points) Perform each indicated operation or explain why it is impossible.

(a) $$\begin{pmatrix} 4 & -2 & 3 \\ 2 & 5 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -5 \\ 3 & 2 & 6 \end{pmatrix}$$

answer: $$\begin{pmatrix} 4 & -1 & -2 \\ 5 & 7 & 5 \end{pmatrix}$$

(b) $$\begin{pmatrix} 5 & -2 \\ 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

answer: $$\begin{pmatrix} 4 \\ 7 \\ 11 \end{pmatrix}$$

(c) $$4 \begin{pmatrix} 2 & 4 \\ 3 & 1 \\ 1 & 5 \end{pmatrix}$$

answer: $$\begin{pmatrix} 8 & 16 \\ 12 & 4 \\ 4 & 20 \end{pmatrix}$$

(d) $$\begin{pmatrix} 4 & -2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

answer: Impossible since the matrix on the left has 3 columns and the matrix on the right only has 2 rows.

(e) $$\begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 2 & 6 \end{pmatrix}$$

answer: $$\begin{pmatrix} 12 & 8 & 24 \\ 6 & 4 & 12 \end{pmatrix}$$

4. (10 points) Find the canonical representative of the matrix-equivalence class (i.e., both row and column operations are allowed) of the following matrix.

$$\begin{pmatrix} 0 & 1 & 3 & 5 \\ 1 & 1 & 1 & 2 \\ 3 & 3 & 3 & 6 \end{pmatrix}$$

answer: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

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5. (10 points) Consider the map

$$\begin{aligned} h : \mathcal{P}_2 &\longrightarrow \mathcal{P}_3 \text{ given by} \\ p(x) &\longmapsto p'(x) + xp(x) \end{aligned}$$

where $p'(x)$ is the derivative of $p(x)$. Find the matrix representing this map with respect to the bases

$$B = \langle 1, x, x^2 \rangle \text{ and } D = \langle 1, x, x^2, x^3 \rangle$$

answer:

$$\begin{aligned} 1 &\longmapsto x \\ x &\longmapsto 1 + x^2 \\ x^2 &\longmapsto 2x + x^3 \end{aligned}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

6. (20 points) Find the inverse of each of the following matrices or explain why the inverse doesn't exist.

(a) $A = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$

answer:

$$A^{-1} = \begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix} \cdot \frac{1}{2} = \begin{pmatrix} 2 & -1 \\ -5/2 & 3/2 \end{pmatrix}$$

(b) $B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$

answer:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

7. (15 points) Use the Gram-Schmidt process to find an orthogonal basis for the vector space

$$M = \text{span}\langle \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix} \rangle$$

answer: Only the second vector needs adjusting. The new vector is

$$\begin{aligned} \vec{v} &= \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \frac{2 \cdot 5 + 1 \cdot 4 + 2 \cdot 2}{2 \cdot 2 + 1 \cdot 1 + 2 \cdot 2} \\ &= \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \frac{18}{9} \\ &= \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \end{aligned}$$

The orthogonal basis is:

$$\langle \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \rangle$$

8. **Extra Credit** (10 points) Project the vector $\vec{v} = \begin{pmatrix} 5 \\ 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}$ orthogonally into the vector space $M = \text{span}\langle \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \\ 1 \end{pmatrix} \rangle$

answer: Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 1 & 1 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}$ then the projected vector is

$$\begin{aligned} A(A^tA)^{-1}A^t\vec{v} &= \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 1 & 1 \\ -1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 8 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 12 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 1 & 1 \\ -1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 8 & -4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 12 \end{pmatrix} \cdot \frac{1}{8} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 1 & 1 \\ -1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -24 \\ 24 \end{pmatrix} \cdot \frac{1}{8} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 1 & 1 \\ -1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot 3 \\ &= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot 3 = \begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \\ 3 \end{pmatrix} \end{aligned}$$