# 1 Quiz (Jan 18)

**Section One.I.2.18c**
Solve the given system using matrix notation. Give the solution set in vector notation.

$$\begin{array}{rcrcrcr} x_1 & & & + & x_3 & = & 4 \\ x_1 & - & x_2 & + & 2x_3 & = & 5 \\ 4x_1 & - & x_2 & + & 5x_3 & = & 17 \end{array}$$

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**Answer:** Form the augmented matrix and put it in reduced echelon form.

$$\left(\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 1 & -1 & 2 & 5 \\ 4 & -1 & 5 & 17 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Then $x$ and $y$ are the leading variables, and $z$ is the free variable. The new system is

$$\begin{array}{rcrcrcr} x_1 & & & + & x_3 & = & 4 \\ & & x_2 & - & x_3 & = & -1 \\ & & & & 0 & = & 0 \end{array}$$

The solution set is

$$\left\{ \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\}$$

## 2 Quiz (Jan 25)

**Section One.II.2.11b**
Find the angle between the following two vectors, if it is defined.

$$v = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix},$$

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**Answer:** We will need the following information:

$$\begin{aligned} v \cdot w &= (1)(0) + (2)(4) + (0)(1) \\ &= 8 \\ \|v\| &= \sqrt{(1)(1) + (2)(2) + (0)(0)} \\ &= \sqrt{5} \\ \|w\| &= \sqrt{(0)(0) + (4)(4) + (1)(1)} \\ &= \sqrt{17} \end{aligned}$$

The answer is the angle

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{v \cdot w}{\|v\|\,\|w\|}\right) \\ &= \cos^{-1}\left(\frac{8}{\sqrt{5}\sqrt{17}}\right) \\ &= \cos^{-1}\left(\frac{8}{\sqrt{85}}\right) \\ &= 0.52019588578126996 \text{ radians} \\ &= 29.805028775336197 \text{ degrees} \end{aligned}$$

# 3 Quiz (Feb 1)

**Section One.III.1.11d**
List all the reduced echelon forms possible for a $3 \times 3$ matrix.

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**Answer:** There are 8 families of these matrices. Each family is determined by which columns have leading 1s. Four of the families have a single member, the rest have and infinite number of members. In the list given below $a, b$ are arbitrary numbers.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{no leading 1s}$$

$$\begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{a single leading 1; in the first column}$$

$$\begin{pmatrix} 0 & 1 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{a single leading 1; in the second column}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{a single leading 1; in the third column}$$

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{two leading 1s; in the first and second columns}$$

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{two leading 1s; in the first and third columns}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{two leading 1s; in the second and third columns}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{three leading 1s; one in each column}$$

## 4 Quiz (Feb 8)

**Section Two.II.1.26**

1. When is a one element set linearly independent?
2. How about a set with two elements?

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**Answer:**

1. Precisely when the element is nonzero.

   To see this suppose that $S = \{\vec{v}\}$ and $\vec{v} \neq \vec{0}$. Then $c\,\vec{v} = \vec{0}$ implies that $c = 0$. On the other hand if $c \neq 0$ and $c\,\vec{v} = \vec{0}$, then (dividing by $c$) we see that $\vec{v} = \vec{0}$.

2. Precisely when neither element is a multiple of the other.

   It is easier, in this case, to show that the set is linearly dependent precisely when one of the elements is a multiple of the other.

   To see this suppose that $S = \{\vec{v_1}, \vec{v_2}\}$. And suppose, for example, $\vec{v_2} = c_1\,\vec{v_1}$. Then $c_1\,\vec{v_1} + c_2\,\vec{v_2} = \vec{0}$ where $c_2 = -1 \neq 0$, so $S$ is linearly dependent. On the other hand suppose that $S$ is linearly dependent. That is $c_1\,\vec{v_1} + c_2\,\vec{v_2} = \vec{0}$ where, say, $c_2 \neq 0$. Then $\vec{v_2} = (-c_1/c_2)\,\vec{v_1}$.

# 5 Quiz (Feb 15)

**Section Two.III.3.40**

1. Find a pair of $2 \times 2$ matrices $A$ and $B$ (both nonzero) with $\text{rank}(A+B) = \text{rank}(A) + \text{rank}(B)$.

2. Find a pair of $2 \times 2$ matrices $A$ and $B$ (both nonzero) with $\text{rank}(A+B) \neq \text{rank}(A) + \text{rank}(B)$.

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**Answer:**

1. There are lots of examples. Here's one:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{rank}(A) = \text{rank}(B) = 1 \text{ and } \text{rank}(A+B) = 2.$$

2. There are lots of examples. Here's one:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad A + B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\text{rank}(A) = \text{rank}(B) = \text{rank}(A+B) = 1.$$

Notice that 5 of my matrices are already in reduced echelon form (and the sixth – $B$ from part (1) – just requires a row swap) so I don't have to do any messy computations to determine their ranks. When you're constructing examples try to find ones that require as little computation as possible.

## 6 Quiz (Feb 22)

**Section Three.I.1.31 (d)** Let $f$ be an automorphism of $\mathbb{R}^2$ with

$$f(\begin{pmatrix}1\\3\end{pmatrix}) = \begin{pmatrix}2\\-1\end{pmatrix} \quad \text{and} \quad f(\begin{pmatrix}1\\4\end{pmatrix}) = \begin{pmatrix}0\\1\end{pmatrix}.$$

Find

$$f(\begin{pmatrix}0\\-1\end{pmatrix}).$$

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**Answer:**

$$\begin{aligned}
f(\begin{pmatrix}0\\-1\end{pmatrix}) &= f(\begin{pmatrix}1\\3\end{pmatrix} - \begin{pmatrix}1\\4\end{pmatrix}) \\
&= f(\begin{pmatrix}1\\3\end{pmatrix}) - f(\begin{pmatrix}1\\4\end{pmatrix}) \\
&= \begin{pmatrix}2\\-1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} \\
&= \begin{pmatrix}2\\-2\end{pmatrix}.
\end{aligned}$$

## 7 Quiz (Feb 29)

**Section Three.III.1.17 (b)** Consider the linear map

$$f : \mathcal{P}^4 \to \mathcal{P}^4 \quad \text{given by}$$
$$p(x) \mapsto p'(x) + \int_0^1 p(x)\,dx$$

Represent the map as a matrix with respect to the basis $B = \langle 1, x, x^2, x^3, x^4 \rangle$. Observe that you are to use the same basis for both the domain and codomain.

**Extra Credit**

(a) What is the nullity (dimension of the kernel) of $f$?

(b) What is the rank (dimension of the image) of $f$?

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**Answer:**

$$\begin{aligned} f(1) &= 1 \\ f(x) &= 3/2 \\ f(x^2) &= 2x + 1/3 \\ f(x^3) &= 3x^2 + 1/4 \\ f(x^4) &= 4x^3 + 1/5 \end{aligned}$$

So we have the matrix:

$$\text{Rep}_{B,B} = \begin{pmatrix} 1 & 3/2 & 1/3 & 1/4 & 1/5 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**Extra Credit**
Looking at the matrix we see that after row reduction we have

$$\begin{pmatrix} 1 & 3/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

There are 4 leading ones (so the rank is 4) and just 1 column without a leading one (so the nullity is 1).

## 8 Quiz (Mar 14)

**Section Three.IV.3.41**
Give an example of two matrices of the same rank with squares of differing rank.

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**Answer:** There are lots of examples. Here's one:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\operatorname{rank}(A) = \operatorname{rank}(B) = 1 \text{ and } \operatorname{rank}(A^2) = 1 \text{ while } \operatorname{rank}(B^2) = 0$$

# 9 Quiz (Mar 21)

**Section Four.V.2.11**
Find the canonical representative of the matrix-equivalence class of each matrix.

(a) $\begin{pmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & 4 \\ 3 & 3 & 3 & -1 \end{pmatrix}$

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**Answer:**

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

# 10 Quiz (Mar 28)

**Section Three.VI.3.1(c)**

Project the vector $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ into

$$M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x + y = 0 \right\}$$

along

$$N = \left\{ c \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; c \in \mathbb{R} \right\}$$

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**Answer:** Since

$$\begin{aligned} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} \\ &= m + n \end{aligned}$$

the projection is

$$\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$$

## 11 Quiz (Apr 4)

**Section Four.I.2.13**
Show that the determinant of the following matrix is zero.

$$\begin{pmatrix} y+z & x+z & x+y \\ x & y & z \\ 1 & 1 & 1 \end{pmatrix}$$

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**Answer:** The following row operations are *shear* opearations, so the determinant isn't changed. The last matrix has a row of zeros so its determinant is zero.

$$\begin{aligned} \begin{pmatrix} y+z & x+z & x+y \\ x & y & z \\ 1 & 1 & 1 \end{pmatrix} &\sim \begin{pmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{pmatrix} && \text{add row 2 to row 1} \\ &\sim \begin{pmatrix} 0 & 0 & 0 \\ x & y & z \\ 1 & 1 & 1 \end{pmatrix} && \text{subtract } (x+y+z) \text{ times row 3 from row 1} \end{aligned}$$

The final matrix

1. Has the same determinant as the first matrix since we only used *shear* operations — and these don't change the determinant.
2. Has determinant equal to zero since it has a row of zeros.

## 12 Quiz (Apr 11)

**Section Four.II.1.12(c)**
By what factor does the following transformation change the size of boxes?

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x - y \\ x + y + z \\ y - 2z \end{pmatrix}$$

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**Answer:** All we need to do is find the determinant of the matrix

$$\begin{aligned}
\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix} &\sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \quad \text{subtract row 1 from row 2} \\
&\sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -5/2 \end{pmatrix} \quad \text{subtract (1/2) times row 2 from row 3}
\end{aligned}$$

So the determinant is $-5$. The map changes the size of blocks by a factor of 5, and also reverses the orientation.

# 13 Quiz (Apr 18)

**Section Five.II.2.7(b)**
Find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$ where

$$\begin{pmatrix} 5 & 4 \\ 0 & 1 \end{pmatrix}$$

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**Answer:** The matrix $A$ is triangular, so its eigenvalues are on the diagonal: $\lambda = 5, 1$. These will be the diagonal entries of $D$. We proceed to find eigenvectors. These will be the columns of $P$.

$$\begin{aligned} \lambda = 5 \qquad & A - 5I = \begin{pmatrix} 0 & 4 \\ 0 & -4 \end{pmatrix} \\ & \mathrm{Nul}(A - 5I) = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle \\ \lambda = 1 \qquad & A - 1I = \begin{pmatrix} 4 & 4 \\ 0 & 0 \end{pmatrix} \\ & \mathrm{Nul}(A - 1I) = \langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle \end{aligned}$$

So we can choose

$$P = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$$

or

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$