

2.7 Related Rates:

Example: A rocket is launched on a vertical trajectory and is tracked by a radar station that is 3 km from the launch pad. Find the vertical speed of the rocket at the instant when the distance from the rocket to the radar station is 5 km and that distance is increasing at 5,000 km/h.

Solution: Step 1: Draw a picture. Include only the information that is constant and not anything that is only true for an instant. For example the 3 km between the launch pad and the station is constant but the 5 km from the station to the rocket is not constant and only true for an instant and should not be labeled on the picture.

Step 2. Choose the variables. For related rate problems, the variables are those which are changing and we either know or want to know their rate of change. So we look through the statement of the problem searching for rates, looking for words like “speed,” or “velocity” or “rate” itself. We can also check the units: if the units are something per unit time then that is a rate. In this problem we see the word “increasing” as well as the units km/h. (kilometers per hour) and this corresponds to the distance from the station to the rocket and so one of our variables is $z = z(t)$ the distance of the rocket to the station. We know $z' = 5000$ at least for an instant. The other rate mentioned is the vertical speed of the rocket. So we make the distance from the rocket to the launch pad another variable say $y = y(t)$. We want $y'(t)$.

Step 3. Relate z and y . By the Pythagorean theorem $3^2 + y^2 = z^2$.

Step 4. Differentiate in t . (The differentiation in all these related rates problems is with respect to time.)

$$0 + 2yy' = 2zz'$$

Step 5. Plug in and answer the question. Now that the differentiation is complete we can substitute the information about the particular instant of interest. We are interested when $z = 5$ and $z' = 5000$. Substitute those values into the equation for the unknown y' : $2yy' = 2(5)(5000)$ or $yy' = 25,000$. What is y ? By Step 3, we have $9 + y^2 = z^2 = 25$ so that $y^2 = 16$ or $y = \pm 4$ and of course $y = 4$ because $y > 0$ for physical reasons. Therefore $4y' = 25,000$ or $y' = 6250$ which says that the vertical velocity of the rocket is 6250 km/h.

Example A trough is 15 feet long and its cross section is an isosceles triangle. The trough is four feet wide at the top and three feet deep. Water is flowing into the trough at 7 cubic feet per minute. How fast is the depth of the water increasing when the depth is 1 foot?

Solution: Draw a picture of the trough with water in it. Label the dimensions of the trough but not of the water.

Choose variables. Here we are told that the water is being pumped into the water at 7 cubic feet per minute and this is a rate of change of volume of water in the trough. Let $V(t)$ denote the volume of water in the trough: we know $dV/dt = 7$. We want to know how fast the level of the water is rising and so we let the depth of the water be $h(t)$.

Relate V and h . Volume is the length of the trough times the cross sectional area: $15\frac{1}{2}bh$. But what is b ? By similar triangles the ratios $b/h = 4/3$ so that

$$V = 15\frac{1}{2}\frac{4}{3}h^2 = 10h^2$$

Differentiate in time

$$\frac{dV}{dt} = 10(2h\frac{dh}{dt}) = 20h\frac{dh}{dt}$$

Substitute $h = 1$ and $dV/dt = 7$

$$7 = 20\frac{dh}{dt}$$

so that the water level is rising at $7/20$ feet per minute at that instant.

Example: Sulphur powder is unloaded from a ship at $3 \text{ m}^3/\text{min}$ by conveyor belt that dumps the sulphur into a conical pile. The height of the pile is always $2/3$ of the diameter of the base. At what rate is the height of the pile increasing when it is 5 meters high?

Solution Draw a picture of a cone with the apex up.

Choose variables. Let $V(t)$ be the volume of the cone. We know that $dV/dt = 3$. Let $h(t)$ be the height of the cone. We want dh/dt .

Relate V and h . The formula for the volume of a cone is

$$V = \frac{\pi}{3}r^2h$$

but $h = (2/3)d = (4/3)r$ so that

$$V = \frac{\pi}{3}\left(\frac{3}{4}h\right)^2h = \frac{3\pi}{16}h^3$$

Differentiate in time

$$V' = \frac{3\pi}{16}3h^2h'$$

so that

$$3 = \frac{9\pi}{16}h^2h' = \frac{9\pi}{16}5^2h' \quad \text{or} \quad h' = \frac{16}{75\pi} \approx 0.068$$

in m/min.