Chapter 5. Exponential, Logarithmic & Inverse Trig Functions

5.1 Inverse Functions:
Examples:

1. If $y = 2x - 4$ then $x = \frac{1}{2}y + 2$. In words: the inverse of the function $f(x) = 2x - 4$ is $f^{-1}(x) = \frac{1}{2}x + 2$. (NOTE: $f^{-1}(x) \neq 1/f(x)$.)

2. If $y = x^2 + 4$ then $x = \sqrt{y-4}$ or $x = -\sqrt{y-4}$ so x cannot be recovered from y. In words: there is no inverse of the function $g(x) = x^2 + 4$.

Definition: A function $f(x)$ with domain A is said to be *one-to-one* if, for any $x_1, x_2 \in A$, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

The function of Example 1 is one-to-one; that of Example 2 is not $(g(-1) = g(1))$.

Definition: Suppose $f(x)$ is a one-to-one function with domain A and range B. Then the inverse f^{-1} of f is the function with domain B and range A defined by $f^{-1}(y) = x$ if and only if $f(x) = y$.

Horizontal Line Test: A function $f(x)$ is one-to-one (so f^{-1} exists) if and only if every horizontal line intersects the graph $y = f(x)$ at most once.

(Recall that the Vertical Line Test was a test fro functions.)

Example 1: $y = 2x + 4$

Example 2: $y = x^2 + 4$, $x \geq 0$.

Finding f^{-1}:

1. Write $y = f(x)$

2. Solve for x in terms of y (if possible).

3. Interchange x and y.

Example: If $f(x) = x^2+4$, $x \geq 0$ then 1. $y = x^2+4$; 2. $x = \sqrt{y-4}$; 3. $f^{-1} = \sqrt{x-4}$.
Graphs: For $y = x^2 + 4$

RULE: If $f(x)$ is one-to-one then the graph of $f^{-1}(x)$ is the reflection of the graph of f in the line $y = x$.

Theorem: Assume $f(x)$ is one-to-one. If f is continuous then f^{-1} is continuous. If f is differentiable at x and $f'(x) \neq 0$ and then $f^{-1}(x)$ is differentiable at $y = f(x)$ and

$$(f^{-1})'(y) = \frac{1}{f'(x)} \text{ where } y = f(x).$$

Example: If $f(x) = x^2 + 4$, $x \geq 0$ then $f'(x) = 2x$. Since $x = f^{-1}(y) = (y-4)^{1/2}$ we have

$$(f^{-1})'(y) = \frac{1}{2x} = \frac{1}{2(y-4)^{1/2}} = \frac{1}{2}(y-4)^{-1/2}$$

except when $y = 4$ which corresponds to $x = 0$.

Showing f is one-to-one: (f is continuously differentiable)

1. If $f'(x) > 0$, $a < x < b$, then f is one to one on $[a, b]$;
2. If $f'(x) < 0$, $a < x < b$, then f is one to one on $[a, b]$.

Verification: By the Mean Value Theorem: If $x_1 < x_2$ there is c, $x_1 < c < x_2$ so that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Therefore $x_2 \neq x_1$ implies $f(x_1) \neq f(x_1)$ because $f'(c) \neq 0$.

Example: Consider $f(x) = (x+2)^3 - 3$. Then $f'(x) = 3(x+2)^2$. Therfore $f(x) > 0$ except when $x = -2$. Therefore $f(x)$ is one to one if restricted to $x < -2$ or to $x > -2$.

On the other hand we can actually find f^{-1}. Solve for x: $y = (x+2)^3 - 3$ means $x = (y+3)^{1/3} - 2$ so that $f^{-1}(x) = (x+3)^{1/3} - 2$. This says that f is one to one on the entire real line and has an inverse. Our earlier theorem says that f^{-1} is differentiable everywhere except possibly at $f(-2) = -3$. We can see in this case that the graph of f^{-1} is vertical at (-3,2).