1. Find the curve's **unit** tangent vector. Also, find the length of the indicated portion of the curve. Show your work. (8 points)

$$\mathbf{r}(t) = (6\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} + (5t)\mathbf{k}, \qquad 0 \le t \le \pi$$

$r'(t) = (12\cos 2t)i - (12\sin 2t)j + 5K$

$|r'(t)| = \sqrt{(12\cos 2t)^2 + (-12\sin 2t)^2 + 5^2} = \sqrt{144\cos^2 2t + 144\sin^2 2t + 25}$

$= \sqrt{144+25} = \sqrt{169} = 13$

Unit tangent vector $= \dfrac{r'(t)}{|r'(t)|} = \boxed{\dfrac{(12\cos 2t)}{13}i - \dfrac{(12\sin 2t)}{13}j + \dfrac{5}{13}K}$

Length $= \displaystyle\int_0^{\pi} |r'(t)|\,dt = \int_0^{\pi} 13\,dt = \boxed{13\pi}$

2. Find the limit. Show your work. (5 points)

$$\lim_{\substack{(x,y)\to(5,2)\\ x-2y\neq 1}} \frac{\sqrt{x-2y}-1}{x-2y-1}$$

$= \displaystyle\lim_{(x,y)\to(5,2)} \frac{(\sqrt{x-2y}-1)(\sqrt{x-2y}+1)}{(x-2y-1)(\sqrt{x-2y}+1)}$

$= \displaystyle\lim_{(x,y)\to(5,2)} \frac{\cancel{(x-2y-1)}}{\cancel{(x-2y-1)}(\sqrt{x-2y}+1)}$

$= \dfrac{1}{\sqrt{5-2\cdot 2}+1} = \dfrac{1}{1+1} = \boxed{\dfrac{1}{2}}$

3. Find $\partial z/\partial u$ when $u = 0, v = 1$ if $z = \sin xy + x \sin y, x = u^2 + v^2, y = uv$. Show your work. (5 points)

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= (y\cos xy + \sin y).\,2u + (x\cos xy + x\cos y).\,v$$

When $u = 0, v = 1$

$x = 0^2 + 1^2 = 1$

$y = 0 \cdot 1 = 0$

$$\therefore \frac{\partial z}{\partial u} = (0\cdot\cos 0 + \sin 0)\cdot 2\cdot 0 + (1\cdot\cos 0 + 1\cos 0)\cdot 1$$

$$= 0 + (2)\cdot 1 = \boxed{2}$$

4. Find the equations for the (a) tangent plane and (b) normal line at the point P_0 on the given surface. Show your work. (8 points)

$$x^2 - xy - y^2 - z = 0, \qquad P_0(1, 1, -1)$$

$f(x,y,z) = x^2 - xy - y^2 - z$

$f_x(x,y,z) = 2x - y$, $f_x(1,1,-1) = 2\cdot 1 - 1 = 1$

$f_y(x,y,z) = -x - 2y$, $f_y(1,1,-1) = -1 - 2\cdot 1 = -3$

$f_z(x,y,z) = -1$, $f_z(1,1,-1) = -1$

Tangent plane

$1.(x-1) - 3(y-1) - 1(z+1) = 0$

$x - 1 - 3y + 3 - z - 1 = 0$

$\boxed{x - 3y - z = -1}$

Normal line

$\boxed{x = 1 + t, \quad y = 1 - 3t, \quad z = -1 - t}$

5. Find all the local maxima, local minima, and saddle points of the function. Show your work. (8 points)

$$f(x, y) = \ln(x + y) + x^2 - y$$

$f_x(x,y) = \dfrac{1}{x+y} + 2x$ $\qquad$ $\dfrac{1}{x+y} + 2x = 0$ — (1)

$f_y(x,y) = \dfrac{1}{x+y} - 1$ $\qquad$ $\dfrac{1}{x+y} - 1 = 0$ — (2)

From (2) $x+y = 1$

$\therefore$ In (1) $\dfrac{1}{1} + 2x = 0 \Rightarrow x = -\dfrac{1}{2}$

$\therefore y = 1 - x = 1 + \dfrac{1}{2} = \dfrac{3}{2}$

<u>Second derivative Test</u>

$f_{xx} = -\dfrac{1}{(x+y)^2} + 2$, $\quad f_{xx}\left(-\frac{1}{2}, \frac{3}{2}\right) = -\dfrac{1}{1} + 2 = 1$

$f_{yy} = -\dfrac{1}{(x+y)^2}$, $\quad f_{yy}\left(-\frac{1}{2}, \frac{3}{2}\right) = -\dfrac{1}{1} = -1$

$f_{xy} = -\dfrac{1}{(x+y)^2}$, $\quad f_{xy}\left(-\frac{1}{2}, \frac{3}{2}\right) = -\dfrac{1}{1} = -1$

$\therefore f_{xx} \cdot f_{yy} - (f_{xy})^2 = 1 \cdot (-1) - (-1)^2 = -1 - 1 = -2 < 0$

$\therefore$ $\boxed{\left(-\frac{1}{2}, \frac{3}{2}\right) \text{ is a saddle point}}$

6. Using the method of Lagrange multipliers, find the point on the surface $z = xy + 1$ nearest to the origin. Show your work. (8 points)

$f(x,y,z) = \sqrt{x^2+y^2+z^2}$ (Distance of any point (x,y,z) on the surface from the origin)

$g(x,y,z) = xy+1-z = 0$

We will minimize $f(x,y,z) = x^2+y^2+z^2$

$\nabla f(x,y,z) = 2xi + 2yj + 2zK$

$\nabla g(x,y,z) = yi + xj - K$

$\nabla f = \lambda \nabla g$ gives.

$2xi + 2yj + 2zK = \lambda y i + \lambda x j - \lambda K$.

$\therefore$ $2x = \lambda y$ — (1)

$2y = \lambda x$ — (2)

$2z = -\lambda$ — (3)

From (1) $x = \frac{\lambda y}{2}$

Putting in (2) $2y = \lambda . \frac{\lambda y}{2} = \frac{\lambda^2 y}{2}$

$\Rightarrow 4y = \lambda^2 y$

$\Rightarrow (\lambda^2 - 4)y = 0$

$\Rightarrow y = 0, \lambda = \pm 2$

$\therefore x = \frac{\lambda y}{2} = 0$

Putting the values of x and y on $g(x,y,z) = 0$ we get

$0.0 + 1 - z = 0$

$z = 1$

$\therefore$ The point is $\boxed{(0,0,1)}$

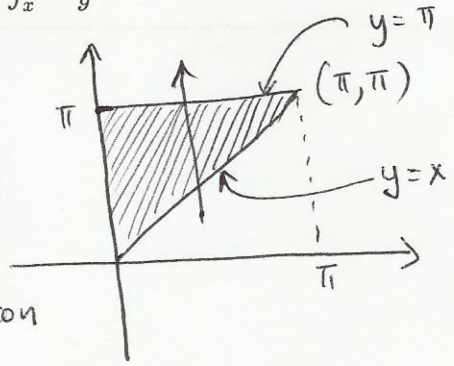
7. Sketch the region of integration, reverse the order of integration, and evaluate the integral. Show your work. (8 points)

$$\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$$

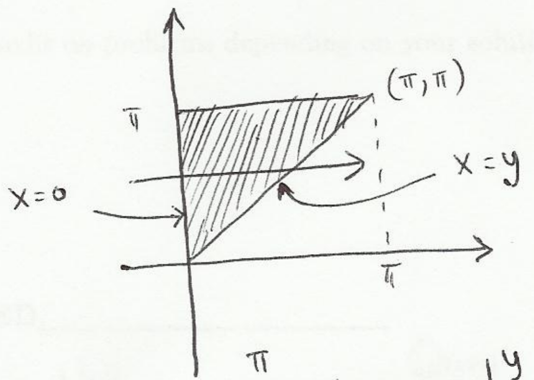
$x \le y \le \pi$

$0 \le x \le \pi$

Since the limits of y involve functions of x therefore the limits are using vertical direction



To reverse the order of integration we now use the horizontal direction



$0 \le x \le y$

$0 \le y \le \pi$

New integration

$$\int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\, dx\, dy = \int_0^{\pi} \frac{\sin y}{y} \cdot x \Big|_0^{y} dy$$

$$= \int_0^{\pi} \frac{\sin y}{\cancel{y}} \cdot \cancel{y}\, dy = \int_0^{\pi} \sin y\, dy$$

$$= -\cos y \Big|_0^{\pi}$$

$$= -\cos \pi + \cos 0$$

$$= -(-1) + 1 = \boxed{2}$$