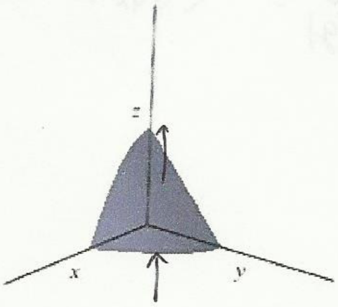
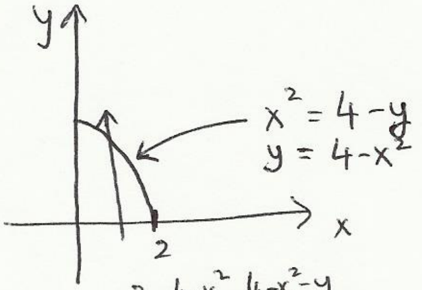
1. Find the volume of the region in the first octant bounded by the coordinate planes and the surface $z = 4 - x^2 - y$. Show your work. (8 points)



Projection on the x-y plane $\qquad 0 \le z \le 4 - x^2 - y$.



$0 \le y \le 4 - x^2$

$0 \le x \le 2$

$$\text{Volume} = \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} dz\,dy\,dx$$

$$= \int_0^2 \int_0^{4-x^2} (4 - x^2 - y)\,dy\,dx$$

$$= \int_0^2 4y - x^2 y - \frac{y^2}{2} \Big|_0^{4-x^2} dx$$

$$= \int_0^2 4(4-x^2) - x^2(4-x^2) - \frac{(4-x^2)^2}{2}\,dx$$

$$= \int_0^2 \left(16 - 4x^2 - \cancel{4x^2} + x^4 - 8 + \cancel{4x^2} - \frac{x^4}{2}\right) dx$$

$$= \int_0^2 \left(8 - 4x^2 + \frac{x^4}{2}\right) dx = 8x - \frac{4x^3}{3} + \frac{x^5}{10} \Big|_0^2 = \left(16 - \frac{32}{3} + \frac{32}{10}\right)$$

$$= \frac{480 - 320 + 96}{30}$$

$$= \boxed{\frac{256}{30}}$$

2. ~~3~~. Using spherical coordinates, find the volume of the region cut from the solid sphere $\rho \le a$ by the half-planes $\theta = 0$ and $\theta = \pi/6$ in the first octant. Show your work. (5 points)

$0 \le \rho \le a$

$0 \le \theta \le \frac{\pi}{6}$

$0 \le \phi \le \frac{\pi}{2}$

$$\text{Volume} = \int_0^{\pi/6}\int_0^{\pi/2}\int_0^a \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\pi/6}\int_0^{\pi/2} \frac{\rho^3}{3} \sin\phi \Big|_0^a \, d\phi \, d\theta$$

$$= -\frac{a^3}{3}\int_0^{\pi/6} \cos\phi \Big|_0^{\pi/2} d\theta$$

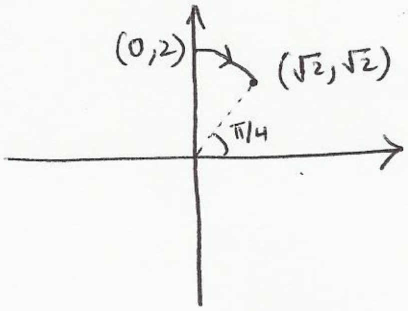
$$= -\frac{a^3}{3}\int_0^{\pi/6} [0-1] \, d\theta = \frac{a^3}{3}\int_0^{\pi/6} d\theta$$

$$= \frac{\pi}{6} \cdot \frac{a^3}{3} = \boxed{\frac{\pi a^3}{18}}$$

3. Evaluate

$$\int_C (x - y)\, ds$$

where $C : x^2 + y^2 = 4$ in the first quadrant from $(0, 2)$ to $(\sqrt{2}, \sqrt{2})$. Show your work. (8 points)



<u>Parametrization</u>

$r(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j}$

$\pi/4 \le t \le \frac{\pi}{2}$ travelling clockwise

$r'(t) = (-2\sin t)\hat{i} + (2\cos t)\hat{j}$

$|r'(t)| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$

$$\int_C (x-y)\, ds = -\int_{\pi/4}^{\pi/2} (2\cos t - 2\sin t) \cdot 2\, dt$$

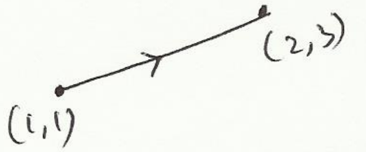
$$= \left[-2\sin t \Big|_{\pi/4}^{\pi/2} - 2\cos t \Big|_{\pi/4}^{\pi/2} \right] \cdot 2$$

$$= \left[-2 + \frac{2}{\sqrt{2}} - \left(2 \cdot 0 - \frac{2}{\sqrt{2}}\right) \right] \cdot 2$$

$$= (2\sqrt{2} - 2) \cdot 2$$

$$= \boxed{4\sqrt{2} - 4}$$

4. ~~5~~. Find the work done by the force $\mathbf{F} = xy\mathbf{i} + (y - x)\mathbf{j}$ over the straight line from (1, 1) to (2, 3). Show your work. (8 points)



Directed vector:

$\langle 1, 2 \rangle$.

$x = 1 + t$ $\quad 0 \le t \le 1$

$y = 1 + 2t$

$r(t) = (1+t)\hat{i} + (1+2t)\hat{j}$

$r'(t) = \hat{i} + 2\hat{j}$

$$\text{Work done} = \int_0^1 F(r(t)) \cdot r'(t)\, dt$$

$$= \int_0^1 \left[(1+t)(1+2t)\hat{i} + t\hat{j}\right] \cdot \left(\hat{i} + 2\hat{j}\right) dt$$

$$= \int_0^1 \left(1 + 3t + 2t^2 + 2t\right) dt$$

$$= \int_0^1 \left(1 + 5t + 2t^2\right) dt$$

$$= t + \frac{5t^2}{2} + \frac{2t^3}{3} \Big|_0^1 = 1 + \frac{5}{2} + \frac{2}{3}$$

$$= \frac{6 + 15 + 4}{6} = \boxed{\frac{25}{6}}$$

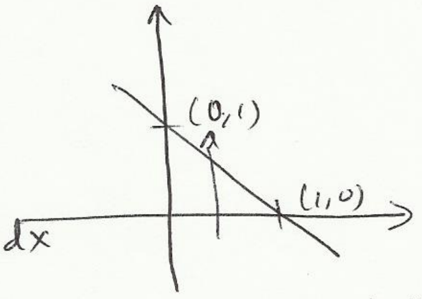
5. ~~6.~~ Apply Green's Theorem to evaluate the integral. Show your work. (8 points)

$$\oint_C (y^2\,dx + x^2\,dy)$$

where C : The triangle bounded by $x = 0, x + y = 1, y = 0$ which is positively oriented.

div $\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$ and Circulation density $\mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$



$0 \le y \le 1-x$

$0 \le x \le 1$

$M = y^2$ $\quad \frac{\partial M}{\partial y} = 2y$

$N = x^2$ $\quad \frac{\partial N}{\partial x} = 2x$

$$= \int_0^1 \int_0^{1-x} (2x - 2y)\,dy\,dx$$

$$= \int_0^1 2xy - y^2 \Big|_0^{1-x} dx$$

$$= \int_0^1 \left[2x(1-x) - (1-x)^2\right] dx$$

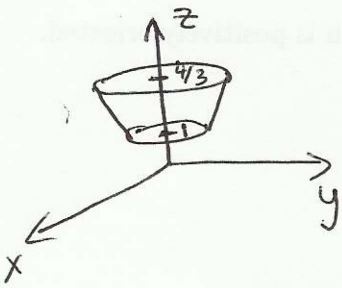
$$= \int_0^1 \left[2x - 2x^2 - 1 + 2x - x^2\right] dx$$

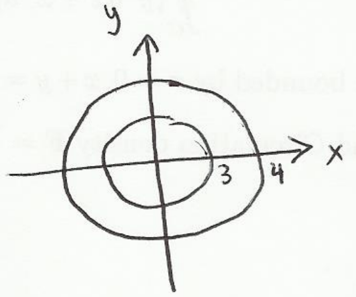
$$= \int_0^1 \left[4x - 3x^2 - 1\right] dx$$

$$= 2x^2 - x^3 - x \Big|_0^1$$

$$= 2 - 1 - 1 = \boxed{0}$$

6. Use a parametrization to express the area of the surface of the portion of the cone $z = \dfrac{\sqrt{x^2+y^2}}{3}$ between the planes $z = 1$ and $z = 4/3$. Then evaluate the integral. Show your work. (10 points)





When $z=1 \Rightarrow 1 = \dfrac{\sqrt{x^2+y^2}}{3}$

$\Rightarrow x^2+y^2 = 9$

When $z = 4/3 \Rightarrow \dfrac{4}{3} = \dfrac{\sqrt{x^2+y^2}}{3}$

$\Rightarrow x^2+y^2 = 16$

Parametrization

$x = r\cos\theta$

$y = r\sin\theta$

$z = \dfrac{\sqrt{x^2+y^2}}{3} = \dfrac{r}{3}$

$3 \le r \le 4$

$0 \le \theta \le 2\pi$

$r(r,\theta) = \langle r\cos\theta, r\sin\theta, \frac{r}{3} \rangle$

$r_r(r,\theta) = \langle \cos\theta, \sin\theta, \frac{1}{3} \rangle$

$r_\theta(r,\theta) = \langle -r\,\cancel{\cos}\sin\theta, r\cos\theta, 0 \rangle$

$$r_r \times r_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & \frac{1}{3} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$= \langle -\frac{1}{3}r\cos\theta, \frac{1}{3}r\sin\theta, r \rangle$

$|r_r \times r_\theta| = \sqrt{\frac{1}{9}r^2\cos^2\theta + \frac{1}{9}r^2\sin^2\theta + r^2}$

$= \sqrt{\frac{1}{9}r^2 + r^2} = \sqrt{\frac{10}{9}r^2} = \frac{r}{3}\sqrt{10}$

Surface Area $= \displaystyle\int_0^{2\pi}\int_3^4 \frac{r}{\cancel{\sqrt{3}}\,3}\sqrt{10}\,dr\,d\theta$

$= \dfrac{\sqrt{10}}{3}\displaystyle\int_0^{2\pi} \frac{r^2}{2}\Big|_3^4\,d\theta$

$= \dfrac{\sqrt{10}}{3}\displaystyle\int_0^{2\pi}\left(8 - \frac{9}{2}\right)d\theta = \frac{\sqrt{10}}{3}\cdot\frac{7}{\cancel{2}}\cdot\cancel{2}\pi = \boxed{\frac{7\sqrt{10}}{3}\pi}$

Bonus Question: Using Green's Theorem, show that the area of the region R bounded by the positively oriented closed curve C is

$$\text{Area of } R = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

By the Flux-Normal form of the Green's Theorem

$$\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$$

$$\therefore \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\iint_R \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}\right) dA$$

$$= \frac{1}{2}\iint 2\,dA$$

$$= \iint dA$$ = Area of the region R.