MATH 2850
Homework 7

Completion points = 10
Selected problem points = 10

Sec 16.4

5 $F = (x-y)i + (y-x)j$ $\quad \frac{\partial M}{\partial x} = 1 \quad \frac{\partial N}{\partial x} = -1$

$M = x-y, \quad N = y-x \quad \frac{\partial N}{\partial y} = 1 \quad \frac{\partial M}{\partial y} = -1$

$$\text{Outward Flux} = \oint_C F\cdot n = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

$$= \int_0^1\int_0^1 (1+1)\,dx\,dy = \boxed{2}$$

$$\text{Circulation} = \oint_C F\cdot T = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_0^1\int_0^1 (-1+1)\,dx\,dy = \boxed{0}$$

25 Area of $R = \frac{1}{2}\oint_C x\,dy - y\,dx$

R: $r(t) = (a\cos t)i + (a\sin t)j \quad 0 \le t \le 2\pi$

$x = a\cos t \quad y = a\sin t$

$dx = -a\sin t\,dt \quad dy = a\cos t\,dt$

$$\frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} (a^2\cos^2 t + a^2\sin^2 t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} a^2\,dt = \boxed{\pi a^2}$$

16.5

19 $z = 2\sqrt{x^2+y^2}$ $x = r\cos\theta,\ y = r\sin\theta$ $0 \le \theta \le 2\pi$

$z = 2\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$

$= 2r$

$2 \le z \le 6 \Rightarrow 2 \le 2r \le 6$

$1 \le r \le 3$

$r(r,\theta) = \langle r\cos\theta, r\sin\theta, 2r\rangle$

$r_r = \langle \cos\theta, \sin\theta, 2\rangle$

$r_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$

$r_r \times r_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 2 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$

$= i(-2r\cos\theta) + j(-2r\sin\theta) + k(r\cos^2\theta + r\sin^2\theta)$

$= \langle -2r\cos\theta, -2r\sin\theta, r\rangle$

$|r_r \times r_\theta| = \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + r^2}$

$= \sqrt{5r^2} = r\sqrt{5}$

$d\sigma = |r_r \times r_\theta|\, dr\, d\theta$

$= r\sqrt{5}\, dr\, d\theta$

Surface Area $= \iint_S |r_r \times r_\theta|\, dr\, d\theta = \int_0^{2\pi}\int_1^3 r\sqrt{5}\, dr\, d\theta = \sqrt{5}\int_0^{2\pi} \frac{r^2}{2}\Big|_1^3 d\theta$

$= 4\sqrt{5} \cdot 2\pi$

$= \boxed{8\sqrt{5}\pi}$

45 $x = 4 - y^2 - z^2$ $y = r\cos\theta,\ z = r\sin\theta$ $0 \le \theta \le 2\pi$

$x = 4 - r^2\cos^2\theta - r^2\sin^2\theta$

$x = 4 - r^2$

$1 \le y^2 + z^2 \le 4$

$1 \le r^2 \le 4$

$1 \le r \le 2$

$$\vec{r}(r,\theta) = \langle 4-r^2,\ r\cos\theta,\ r\sin\theta \rangle$$

$$\vec{r}_r = \langle -2r,\ \cos\theta,\ \sin\theta \rangle$$

$$\vec{r}_\theta = \langle 0,\ -r\sin\theta,\ r\cos\theta \rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ -2r & \cos\theta & \sin\theta \\ 0 & -r\sin\theta & r\cos\theta \end{vmatrix}$$

$$= i\left(r\cos^2\theta + r\sin^2\theta\right) + j\left(2r^2\cos\theta\right) + k\left(2r^2\sin\theta\right)$$

$$= \langle r,\ 2r^2\cos\theta,\ 2r^2\sin\theta \rangle$$

$$\left|\vec{r}_r \times \vec{r}_\theta\right| = \sqrt{r^2 + 4r^4} = r\sqrt{1+4r^2}$$

$$\text{Surface area} = \iint_S \left|\vec{r}_r \times \vec{r}_\theta\right| \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_1^2 r\sqrt{1+4r^2}\, dr\, d\theta$$

$u = 1+4r^2$

$du = 8r\, dr$

$$= \frac{1}{8}\int_0^{2\pi}\int_5^{17} \sqrt{u}\, du\, d\theta$$

$$= \frac{1}{\cancel{8}_4}\int_0^{2\pi} \frac{u^{3/2}}{3}\cdot \cancel{2}\,\Big|_5^{17} d\theta$$

$$= \frac{\pi}{2}\left[\frac{17^{3/2}}{3} - \frac{5^{3/2}}{3}\right]$$

$$= \boxed{\frac{\pi}{6}\left[17^{3/2} - 5^{3/2}\right]}$$

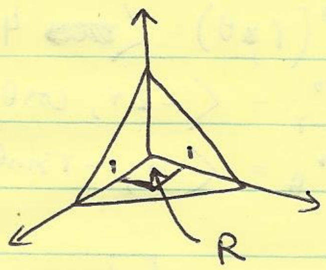
Sec 16.6

5 $F(x,y,z) = z$

$$\underbrace{x+y+z}_{f} = 4$$

$$\nabla f = \langle 1,1,1 \rangle$$

$$\nabla f \cdot k = 1$$



$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot k|} = \frac{|\langle 1,1,1\rangle|}{1} = \sqrt{3}\, dA$$

$$\iint_S z\, d\sigma = \int_0^1\int_0^1 (4-x-y)\sqrt{3}\, dx\, dy$$

$$= \sqrt{3}\int_0^1 4x - \frac{x^2}{2} - yx \Big|_0^1 dy$$

$$= \sqrt{3}\int_0^1 \left[4 - \frac{1}{2} - y\right] dy$$

$$= \sqrt{3}\int_0^1 \left[\frac{7}{2} - y\right] dy = \sqrt{3}\left[\frac{7y}{2} - \frac{y^2}{2}\right]\Big|_0^1$$

$$= \boxed{\sqrt{3}\cdot 3}$$