MATH 2850 Sec 007
ELEMENTARY MULTIVARIABLE CALCULUS
QUIZ 7
November 29, 2012

Name (Last, First) key

$\rho = 1$

1. Consider the unit sphere $x^2 + y^2 + z^2 = 1$. Find $d\sigma$. Show your work.

$x = \sin\phi\cos\theta \qquad 0 \le \phi \le \pi$

$y = \sin\phi\sin\theta \qquad 0 \le \theta \le 2\pi$

$z = \cos\phi$

$r(\phi,\theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$

$r_\phi = \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi \rangle$

$r_\theta = \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0 \rangle$

$$r_\phi \times r_\theta = \begin{vmatrix} i & j & k \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= i\left(\sin^2\phi\cos\theta\right) + j\left(\sin^2\phi\sin\theta\right) + k\left(\sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta\right)$$

$$= \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi \rangle$$

$$d\sigma = |r_\phi \times r_\theta| = \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi}$$

$$= \sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi} = \sqrt{\sin^2\phi}$$

$$= \boxed{\sin\phi}$$

2. Using the above result evaluate

$$\iint_S x^2 \, d\sigma$$

where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

$$\iint_S x^2 d\sigma = \int_0^{2\pi}\int_0^{\pi} \sin^2\phi\cos^2\theta \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^3\phi\cos^2\theta \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} (1-\cos^2\phi)\sin\phi\cos^2\theta \, d\phi \, d\theta$$

$u = \cos\phi$, $du = -\sin\phi \, d\phi$

$$= \int_0^{2\pi} \left(\frac{\cos^3\phi}{3} - \cos\phi\right)\Big|_0^{\pi} \cos^2\theta \, d\theta$$

$$= \int_0^{2\pi} \left(-\frac{1}{3} + 1 - \frac{1}{3} + 1\right)\cos^2\theta \, d\theta$$

$$= \frac{4}{3}\int_0^{2\pi} \cos^2\theta \, d\theta = \frac{2}{3}\int_0^{2\pi} (1+\cos 2\theta)\, d\theta = \frac{2}{3}\left[\theta + \frac{\sin 2\theta}{2}\right]\Big|_0^{2\pi}$$

$$= \boxed{\frac{4\pi}{3}}$$