Second Order Differential Equations
Lecture 6

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6.1. Outline of Lecture

- Repeated Roots; Reduction of Order
- Nonhomogeneous Equations; Method of Undetermined Coefficients
- Variation of Parameters

6.2. Repeated Roots; Reduction of Order

In the previous lectures we looked at second order linear homogeneous equations with constant coefficients whose characteristic equation has either different real roots or complex roots. Now we look into the final case, when the characteristic equation has repeated roots.

The characteristic equation of the second order linear homogeneous equation

$$ay'' + by' + cy = 0. \tag{6.1}$$

is

$$ar^2 + br + cr = 0. \tag{6.2}$$

When the above equation has repeated roots then its discriminant $b^2 - 4ac$ is zero. Then the roots are

$$r_1 = r_2 = -b/2a. \tag{6.3}$$

Both these roots yield the same solution. In this case we use the method due to D'Alembert to find a different solution. Recall that since $y_1(t)$ is a solution of Eq. (6.1), so is cy_1 for any constant c. The basic idea is to generalize this observation by replacing c by a function $v(t)$ and then trying to determine $v(t)$ so that the product $v(t)y_1(t)$ is also a solution of Eq. (6.1). We demonstrate this method using the next example.

Example 1. Solve the differential equation

$$y'' + 6y' + 9y = 0. \tag{6.4}$$

Solution 1. The characteristic equation is

$$r^2 + 6r + 9 = (r+3)^2 = 0. \tag{6.5}$$

so $r_1 = r_2 = -3$. Therefore one solution is $y_1(t) = e^{-3t}$. Let $y = v(t)y_1(t)$. We substitute $y = v(t)y_1(t)$ in Eq. (6.4) and use the resulting equation to find $v(t)$. Starting with

$$y = v(t)y_1(t) = v(t)e^{-3t}. \tag{6.6}$$

we have

$$y' = v'(t)e^{-3t} - 3v(t)e^{-3t}. \tag{6.7}$$

and

$$y'' = v''(t)e^{-3t} - 6v'(t)e^{-3t} + 9v(t)e^{-3t}. \tag{6.8}$$

By substituting the expressions in Eqs. (6.6), (6.7), (6.8) in Eq. (6.4) and collecting terms, we obtain

$$[v''(t) - 6v'(t) + 9v(t) + 6v'(t) - 18v(t) + 9v(t)]e^{-3t} = 0. \tag{6.9}$$

which simplifies to

$$v''(t) = 0. \tag{6.10}$$

Therefore

$$v'(t) = c_1 \tag{6.11}$$

and

$$v(t) = c_1 t + c_2, \tag{6.12}$$

where c_1 and c_2 are arbitrary constants. Finally substituting for $v(t)$ in Eq. (6.6), we obtain

$$y = c_1 te^{-3t} + c_2 e^{-3t}. \tag{6.13}$$

The second term on the right side of Eq. (6.13) corresponds to the original solution $y_1(t) = e^{-3t}$, but the first term arises from a second solution, namely $y_2(t) = te^{-3t}$. We can verify that these solutions form a fundamental set by calculating their Wronskian. The Wronskian turns out to be

$$W(y_1, y_2)(t) = e^{-6t} \neq 0. \tag{6.14}$$

The procedure used in the above example can be generalized to a more general equation whose characteristic equation has repeated roots. In general for an equation

$$ay'' + by' + cy = 0 \tag{6.15}$$

the general solution is

$$y = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}. \tag{6.16}$$

where c_1 and c_2 are arbitrary constants. The geometrical behavior of solutions is similar in this case to that when the roots are real and different. If the exponents are either positive or negative, then the magnitude of the solution grows or decays accordingly; the linear factor t has little significance. However, if the repeated root is zero, then the differential equation is $y'' = 0$ and the general solution is a linear function of t.

6.2.1. Reduction of Order

The method discussed in the earlier section is more generally applicable. Suppose that we know one solution $y_1(t)$, not everywhere zero, of

$$y'' + p(t)y' + q(t)y = 0. \tag{6.17}$$

We can assume the other solution is $v(t)y_1(t)$ and apply the earlier method to find $v(t)$. We illustrate this in the next example.

Example 2. Given that $y_1(t) = t^{-1}$ is a solution of

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0, \tag{6.18}$$

find a fundamental set of solutions.

Solution 2. We set $y = v(t)t^{-1}$, then

$$y' = v't^{-1} - vt^{-2}, \quad y'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}. \tag{6.19}$$

Substituting for y, y', and y'' in Eq. (6.18) and collecting terms, we obtain

$$2t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + 3t(v't^{-1} - vt^{-2}) - vt^{-1} \tag{6.20}$$

$$= 2tv'' - v' = 0. \tag{6.21}$$

Therefore we see that Eq. (6.21) is a separable equation, by noting that $v'' = (v')'$

Separating them out makes both side integrable,

$$\int \frac{(v')'}{v'} = \int \frac{1}{2t}. \tag{6.22}$$

$$\ln |v'(t)| = \ln |ct^{1/2}|. \tag{6.23}$$

Therefore

$$v'(t) = ct^{1/2};$$

then

$$v(t) = \frac{2}{3}ct^{3/2} + k.$$

It follows that

$$y = \frac{2}{3}ct^{1/2} + kt^{-1}. \tag{6.24}$$

where c and k are arbitrary constants. The second term on the right side of Eq. (6.24) is a multiple of $y_1(t)$ and can be dropped, but the first term provides a new solution of $y_2(t) = t^{1/2}$. The Wronskian of y_1 and y_2 is

$$W(y_1, y_2)(t) = \frac{3}{2}t^{-3/2}, \quad t > 0. \tag{6.25}$$

Consequently, y_1 and y_2 form a fundamental set of solutions of Eq. (6.18).

6.3. Nonhomogeneous Equations; Method of Undetermined Coefficients

In this section we learn how to solve a special type of the general nonhomogeneous equation, specifically equations of the form

$$ay'' + by' + cy = g(t), \tag{6.26}$$

where a, b, and c are constants and $g(t)$ is a special function of t.

Before embarking on that, we look at two results that describe the structure of solutions of the general nonhomogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \tag{6.27}$$

where p, q, and g are given continuous functions of the open interval I. Let

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{6.28}$$

be the homogeneous equation corresponding to Eq. (6.27).

Theorem 6.29. *If Y_1 and Y_2 are two solutions of the nonhomogeneous equation (6.27), then their difference $Y_1 - Y_2$ is a solution of the corresponding homogeneous equation (6.28). If in addition, y_1 and y_2 are a fundamental set of solutions of Eq. (6.28), then*

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t), \tag{6.30}$$

where c_1 and c_2 are constants.

Proof of the above theorem follows from previous lectures and simple algebra and can be found in the text book.

Theorem 6.31. *The general solution of the nonhomogeneous equation (6.27) can be written in the form*

$$y = \phi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t), \tag{6.32}$$

where y_1 and y_2 are a fundamental set of solutions of the corresponding homogeneous equation (6.28), c_1 and c_2 are arbitrary constants, and Y is some specific solution of the nonhomogeneous equation (6.27).

The proof of Theorem (6.31) follows quickly from the preceding theorem. We can think of Y_1 as arbitrary solution ϕ and Y_2 as the specific solution Y.

Theorem (6.31) states that to solve the nonhomogeneous equation (6.27), we must do three things:

1. Find the general solution $c_1 y_1(t) + c_2 y_2(t)$ of the corresponding homogeneous equation. This solution is sometimes called the complementary solution and denoted by $y_c(t)$.
2. Find some specific solution $Y(t)$ of the nonhomogeneous equation. This solution is sometimes called the particular solution.
3. Add together the functions found in the two preceding steps.

Since we already know how to find $y_c(t)$, for homogeneous equations with constant coefficients, we would therefore like to find a specific solution of the nonhomogeneous equation (6.26) that we mentioned earlier at the beginning of the section.

We do this in this section for some special functions $g(t)$ in Eq. (6.26) using the Method of Undetermined Coefficients. This method requires us to make an initial assumption about the form of the particular solution $Y(t)$, but with the coefficients left unspecified. We then substitute the assumed expression into the equation and attempt to determine the coefficients so as to satisfy that equation. We summarize the method next.

6.3.1. Method of Undetermined Coefficients.

To find the particular solution let us begin with nonhomogeneous equation with constant coefficients

$$ay'' + by' + cy = g(t), \tag{6.33}$$

where a, b, and c are constants.

1. We make sure that the function $g(t)$ in Eq. (6.26) belongs to one of the classes of functions in the next table, that is, it involves nothing more than exponential functions, sines, cosines, polynomials, or sum or products of such functions.
2. If $g(t) = g_1(t) + \cdots + g_n(t)$, that is, if $g(t)$ is a sum of n terms, then we form n subproblems, each of which contains only one of the terms $g_1(t), \ldots, g_n(t)$. The ith subproblem consists of the equation

$$ay'' + by' + cy = g_i(t), \tag{6.34}$$

   where i runs from 1 to n.
3. Depending on $g_i(t)$, we assume the particular solution $Y_i(t)$ according to the next table.

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$	$A_0t^n + A_1t^{n-1} + \cdots + A_n$
$P_n(t)e^{\alpha t}$	$(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\sin\beta t$ or $P_n(t)e^{\alpha t}\cos\beta t$	$(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}\cos\beta t + (B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t}\sin\beta t$

4. If there is any duplication in the assumed form of $Y_i(t)$ with the solutions of the corresponding homogeneous equation, then multiply $Y_i(t)$ by t, or (if necessary) by t^2, so as to remove the duplication. So for instance if we want to find a particular solution of

$$y'' + 4y' + 4y = 6te^{-2t}, \tag{6.35}$$

   our choice of $Y(t)$ would have to be At^2e^{-2t} since te^{-2t} (which we find from the above table) is a solution of the corresponding homogeneous equation of Eq. (6.35).
5. Find a particular solution $Y_i(t)$ for each subproblems. Then the sum $Y_1(t) + \cdots + Y_n(t)$ is a particular solution of the full nonhomogeneous equation (6.26).

Let us look at an example which uses the above method.

Example 3. Find the particular solution of

$$y'' - 3y' - 4y = 2e^{-t} \tag{6.36}$$

Solution 3. The table says that our assumption for $Y(t)$ should be Ae^{-t} for some constant A that is to be determined. However e^{-t} is a solution of the corresponding homogeneous equation of (6.36)

$$y'' - 3y' - 4y = 0 \tag{6.37}$$

Therefore we modify our assumption of $Y(t)$, by multiplying it with t and assume that the particular solution is of the form $Y(t) = Ate^{-t}$. Then

$$Y'(t) = Ae^{-t} - Ate^{-t}, \quad Y''(t) = -2Ae^{-t} + Ate^{-t}.$$

Substituting these expressions for y, y' and y'' in Eq. (6.36), we obtain

$$(-2A - 3A)e^{-t} + (A + 3A - 4A)te^{-t} = 2e^{-t}. \tag{6.38}$$

Hence $-5A = 2$, so $A = -2/5$. Thus a particular solution of Eq. (6.36) is

$$Y(t) = -\frac{2}{5}te^{-t}. \tag{6.39}$$

6.4. Variation of Parameters

In this section we describe another method of finding a particular solution of a non-homogenoeus equation. This method is known as **variation of parameters**. The main advantage of variation of parameters is that it is a general method. Without further adieu, let's look into the general theorem illustrating the method.

Theorem 6.40. *If the functions p, q and g are continuous on an open interval I, and if the functions y_1 and y_2 are a fundamental set of solutions of the homogeneous equation (6.28) corresponding to the non-homogeneous equation (6.27)*

$$y'' + p(t)y' + q(t)y = g(t), \tag{6.41}$$

then a particular solution of Eq. (6.27) is

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)}\, ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)}\, ds, \tag{6.42}$$

where t_0 is any conveniently chosen point in I. The general solution is

$$y = c_1y_1(t) + c_2y_2(t) + Y(t). \tag{6.43}$$

As was mentioned earlier, this method is a general method; in principle at least, it can be applied to any equation, and it requires no detailed assumptions about the form of the solution. On the other hand, the method of variation of parameters requires us to evaluate certain integrals involving the nonhomogeneous term in the differential equation, and this may present difficulties.

We dive into an example which uses the above method.

Example 4. The given functions y_1 and y_2 satisfy the corresponding homogeneous equation. Find a particular solution of the given nonhomogeneous equation.

$$t^2y'' - 2y = 3t^2 - 1, \quad t > 0, \quad y_1(t) = t^2, \ \ y_2(t) = t^{-1} \tag{6.44}$$

Solution 4. Writing the above equation in the standard form we have,

$$y'' - \frac{2}{t^2}y = 3 - \frac{1}{t^2} \tag{6.45}$$

Therefore $p(t) = 0, q(t) = -\frac{2}{t^2}$ and $g(t) = 3 - \frac{1}{t^2}$. The thress functions are continuous whenever $t \neq 0$. Therefore we choose $t_0 = 1$. We also have $W(y_1, y_2) = -3$. By the above theorem

$$\begin{aligned} Y(t) &= -t^2 \int_1^t \frac{\frac{1}{s} \cdot (3 - \frac{1}{s^2})}{-3} \, ds + \frac{1}{t} \int_1^t \frac{s^2 \cdot (3 - \frac{1}{s^2})}{-3} \, ds \\ &= \frac{t^2}{3} \int_1^t (\frac{3}{s} - \frac{1}{s^3}) \, ds - \frac{1}{3t} \int_1^t (3s^2 - 1) \, ds \end{aligned}$$

Integrating the above expression and using the limits we have

$$Y(t) = \frac{t^2}{3}(3 \ln t + \frac{1}{2t^2} - \frac{1}{2}) - \frac{1}{3t}(t^3 - t)$$

After simplification we have

$$Y(t) = t^2 \ln t + \frac{1}{2} - \frac{t^2}{2}$$

Since t^2 is already a solution of the corresponding homogeneous equation, we can ignore it at this moment. Hence the particular solution of Eq. (6.44) is given by

$$Y(t) = t^2 \ln t + \frac{1}{2} \tag{6.46}$$