1. Find the point (x, y), at which the graph of $y = 3x^2 + 9x - 9$ has a horizontal tangent. Show your work. (6 points)

$$\frac{dy}{dx} = 6x + 9 = 0$$

$$x = -\frac{9}{6} = -\frac{3}{2}$$

$$y = 3\cdot\left(-\frac{3}{2}\right)^2 + 9\left(-\frac{3}{2}\right) - 9$$

$$= 3\cdot\frac{9}{4} - \frac{27}{2} - 9$$

$$= \frac{27}{4} - \frac{27}{2} - 9 = -\frac{27}{4} - 9 = -\frac{63}{4}$$

$\therefore \left(-\frac{3}{2}, -\frac{63}{4}\right)$ is the point where $y = 3x^2 + 9x - 9$ has a horizontal tangent.

2. Find equations of all lines having slope -1 that are tangent to the curve $y = \dfrac{1}{x-3}$. Show your work. (6 points)

$$\frac{dy}{dx} = -\frac{1}{(x-3)^2} = -1$$

$\Rightarrow (x-3)^2 = 1$

$\Rightarrow x - 3 = \pm 1$

$\Rightarrow x = 4, 2$

$\Rightarrow y = \frac{1}{4-3}, \frac{1}{2-3} = 1, -1$

Tangent at $(4,1)$

$y - 1 = -1(x - 4)$

$y - 1 = -x + 4$

$\boxed{y = -x + 5}$

Tangent at $(2,-1)$

$y + 1 = -1(x - 2)$

$y + 1 = -x + 2$

$\boxed{y = -x + 1}$

3. Using the definition, calculate the derivative of the function. Then find the value of the derivative as specified. Show your work. (8 points)

$$g(t) = \frac{3}{t} \quad g'(-2), g'(3), g'(\sqrt{2})$$

$$g(t+h) = \frac{3}{t+h}$$

$$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h} = \lim_{h \to 0} \frac{\frac{3}{t+h} - \frac{3}{t}}{h}$$

$$= \lim_{h \to 0} \frac{3t - 3(t+h)}{(t+h) \cdot t} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3t} - \cancel{3t} - 3h}{t(t+h)} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{-3}{t(t+h)} = \boxed{-\frac{3}{t^2}}$$

$$g'(-2) = -\frac{3}{(-2)^2} = \boxed{-\frac{3}{4}}$$

$$g'(3) = -\frac{3}{3^2} = -\frac{3}{9} = \boxed{-\frac{1}{3}}$$

$$g'(\sqrt{2}) = -\frac{3}{(\sqrt{2})^2} = \boxed{-\frac{3}{2}}$$

4. Determine if the following piecewise defined function is **differentiable** at $x = 1$. Show your work. (8 points)

$$f(x) = \begin{cases} \overset{-3x}{\cancel{3x - 2}}, & x \geq 1 \\ x^2 - 3x - 1, & x < 1 \end{cases}$$

Left hand derivative

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1^-} \frac{x^2 - 3x - 1 - (-3)}{x - 1}$$

$$= \lim_{x \to 1^-} \frac{x^2 - 3x + 2}{x - 1}$$

$$= \lim_{x \to 1^-} \frac{(x-1)(x-2)}{(x-1)} = \lim_{x \to 1^-} (x-2) = -1$$

Right hand derivative

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1^+} \frac{-3x + 3}{x - 1}$$

$$= \lim_{x \to 1^+} \frac{-3(x-1)}{x - 1} = -3$$

As these one-sided derivatives are not equal hence $f(x)$ is not differentiable at $x = 1$.

5. Find all points (x, y) on the graph of $y = \dfrac{x}{x-3}$ with tangent lines perpendicular to the line $y = 3x + 2$ Show your work. (6 points)

$m = 3$, $m_{\perp} = -\frac{1}{3}$

$$\frac{dy}{dx} = \frac{(x-3)\cdot\frac{d}{dx}(x) - x\frac{d}{dx}(x-3)}{(x-3)^2}$$

$$= \frac{(x-3).1 - x.1}{(x-3)^2} = \frac{x-3-x}{(x-3)^2} = -\frac{3}{(x-3)^2}$$

$\therefore$ $\dfrac{-3}{(x-3)^2} = -\dfrac{1}{3}$

$\Rightarrow (x-3)^2 = 9$

$\Rightarrow x - 3 = \pm 3$

$\Rightarrow x = 6, 0$

$\Rightarrow y = \dfrac{6}{6-3}, \dfrac{0}{0-3} = 2, 0$

$\therefore$ The points are $(6, 2)$ and $(0, 0)$.

6. An object is dropped from a tower, 154 ft from the ground. The object's height above ground t sec into the fall is $s = 154 - 16t^2$. Show your work. (8 points)

   (a) What is the object's velocity, speed, and acceleration at time t?
   (b) About how long does it take the object to hit the ground?
   (c) What is the object's velocity at the moment of impact?

(a) $v = \frac{ds}{dt} = -32t$, speed $= |v| = 32t$

$a = \frac{dv}{dt} = -32$ ft/sec^2

(b) $s = 0$

$154 - 16t^2 = 0$

$16t^2 = 154$

$t^2 = \frac{\cancel{154}\ 77}{\cancel{16}\ 8} = 9.625$

$\boxed{t = 3.1 \text{ sec.}}$

(c) $v = -32t = -32 \times 3.1$ ft/sec

$\boxed{= -99.2 \text{ ft/sec.}}$

7. Let $f(x) = 7x^3 - 13x^2 - 4$, $x \geq 1.5$. Find the value of $\dfrac{df^{-1}}{dx}$ at the point $x = 1760 = f(7)$. Show your work. (6 points)

$$\left.\frac{df^{-1}}{dx}\right|_{x=1760} = \frac{1}{f'(f^{-1}(1760))} = \frac{1}{f'(7)}$$

$$f'(x) = 21x^2 - 26x$$

$$f'(7) = 21 \times 49 - 26 \times 7$$

$$= 847$$

$$\boxed{\left.\frac{df^{-1}}{dx}\right|_{x=1760} = \frac{1}{847}}$$

8. Write the function in the form $y = f(u)$ and $u = g(x)$. Then find $\dfrac{dy}{dx}$ as a function of x. Show your work. (6 points)

$$y = \left(2 - \sin^3 x\right)^{-4}$$

$g(x) = 2 - \sin^3 x$ $\qquad$ $g'(x) = -3\sin^2 x \cdot \cos x$

$f(u) = u^{-4}$ $\qquad$ $f'(u) = -4u^{-5}$

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

$$= -4(2 - \sin^3 x)^{-5} \cdot (-3\sin^2 x \cos x)$$

$$= \boxed{\frac{12\sin^2 x \cos x}{(2 - \sin^3 x)^5}}$$

9. Find $\frac{dp}{dq}$. Show your work. (6 points)

$$p = \frac{\sin^2 q - \cos q}{\cos^2 q}$$

$$\frac{dp}{dq} = \frac{\cos^2 q \cdot \frac{d}{dq}(\sin^2 q - \cos q) - (\sin^2 q - \cos q)\frac{d}{dq}\cos^2 q}{\cos^4 q}$$

$$= \frac{\cos^2 q \cdot (2\sin q\cos q + \sin q) + (\sin^2 q - \cos q)\, 2\cos q \sin q}{\cos^4 q}$$

$$= \frac{2\sin q\cos^3 q + \sin q\cos^2 q + 2\cos q\sin^3 q - 2\cos^2 q\sin q}{\cos^4 q}$$

$$= \frac{2\sin q\cos q(\cos^2 q + \sin^2 q) - \cos^2 q\sin q}{\cos^4 q}$$

$$= \frac{2\sin q\cos q - \cos^2 q\sin q}{\cos^4 q} = \boxed{\frac{2\sin q}{\cos^3 q} - \frac{\sin q}{\cos^2 q}}$$

10. Find $\frac{dy}{dt}$. Show your work. (8 points)

$$y = \sin^3(\cos^2 t)$$

$f(x) = \sin^3 x$, $f'(x) = 3\sin^2 x\cos x$

$g(t) = \cos^2 t$, $g'(t) = -2\cos t\sin t$

$y = f(g(t))$

$y' = f'(g(t))\cdot g'(t)$

$= 3\sin^2(\cos^2 t)\cos(\cos^2 t)\cdot(-2\cos t\sin t)$

$= \boxed{-6\sin^2(\cos^2 t)\cos(\cos^2 t)\cos t\sin t}$

11. Find the slope of the tangent and normal line to the curve at the given point. Show your work. (8 points)

$$xy^2 - 3y^2 + x^2y = 18, \quad (3, 2)$$

$$\frac{d}{dx}(xy^2) - \frac{d}{dx}(3y^2) + \frac{d}{dx}(x^2y) = \frac{d}{dx}(18)$$

$$\Rightarrow x \cdot \frac{d}{dx}(y^2) + y^2 \cdot \frac{dx}{dx} - 3 \cdot 2y\frac{dy}{dx} + y\frac{d}{dx}(x^2) + x^2\frac{dy}{dx} = 0$$

$$\Rightarrow 2yx\frac{dy}{dx} + y^2 - 6y\frac{dy}{dx} + 2xy + x^2\frac{dy}{dx} = 0$$

$$\Rightarrow 6y\frac{dy}{dx} - x^2\frac{dy}{dx} - 2xy\frac{dy}{dx} = y^2 + 2xy.$$

$$\Rightarrow \frac{dy}{dx}(6y - x^2 - 2xy) = y^2 + 2xy.$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 + 2xy}{6y - x^2 - 2xy}.$$

$$\Rightarrow \frac{dy}{dx}\bigg|_{(3,2)} = \frac{2^2 + 2.3.2}{6.2 - 3^2 - 2.3.2} = \frac{4 + 12}{12 - 9 - 12} = \boxed{\frac{16}{-9}}$$

Slope of the tangent line

$$\boxed{\text{Slope of the normal line} = \frac{9}{16}}$$

12. Using Logarithmic differentiation, find the derivative of the following function. Show your work. (8 points)

$$f(x) = \frac{(x+6)^{\frac{1}{2}}(x-2)^2}{(x+3)^{\frac{3}{2}}(x-1)}$$

$$\ln f(x) = \ln\left[\frac{(x+6)^{1/2}(x-2)^2}{(x+3)^{3/2}(x-1)}\right]$$

$$\therefore \ln f(x) = \ln(x+6)^{1/2} + \ln(x-2)^2 - \ln(x+3)^{3/2} - \ln(x-1)$$

$$\Rightarrow \ln f(x) = \frac{1}{2}\ln(x+6) + 2\ln(x-2) - \frac{3}{2}\ln(x+3) - \ln(x-1)$$

$$\Rightarrow \frac{1}{f(x)}\cdot f'(x) = \frac{1}{2(x+6)} + \frac{2}{(x-2)} - \frac{3}{2(x+3)} - \frac{1}{(x-1)}$$

$$\Rightarrow f'(x) = f(x)\left[\frac{1}{2(x+6)} + \frac{2}{(x-2)} - \frac{3}{2(x+3)} - \frac{1}{(x-1)}\right]$$

$$\boxed{= \frac{(x+6)^{1/2}(x-2)^2}{(x+3)^{3/2}(x-1)}\left[\frac{1}{2(x+6)} + \frac{2}{(x-2)} - \frac{3}{2(x+3)} - \frac{1}{(x-1)}\right]}$$

13. Use implicit differentiation to find $\frac{dy}{dx}$. Show your work. (8 points)

$$x^2 \sin(4x - 7y^2) = y \cos 2x$$

$$\frac{d}{dx}\left(x^2 \sin(4x-7y^2)\right) = \frac{d}{dx}(y\cos 2x)$$

$$\Rightarrow \quad x^2 \frac{d}{dx}\left(\sin(4x-7y^2)\right) + \sin(4x-7y^2) \cdot \frac{d}{dx}(x^2) = y\frac{d(\cos 2x)}{dx} + \cos 2x \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad x^2 \cdot \cos(4x-7y^2) \cdot \frac{d}{dx}(4x-7y^2) + 2x\sin(4x-7y^2) = -2y\sin 2x + \cos 2x \frac{dy}{dx}$$

$$\Rightarrow \quad x^2 \cdot \cos(4x-7y^2) \cdot \left(4 - 14y\frac{dy}{dx}\right) + 2x\sin(4x-7y^2) = -2y\sin 2x + \cos 2x \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad 4x^2\cos(4x-7y^2) - 14x^2y\cos(4x-7y^2) \cdot \frac{dy}{dx} + 2x\sin(4x-7y^2) = -2y\sin 2x + \cos 2x \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad 14x^2y\cos(4x-7y^2)\frac{dy}{dx} + \cos 2x\frac{dy}{dx} = 4x^2\cos(4x-7y^2) + 2x\sin(4x-7y^2) + 2y\sin 2x$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{4x^2\cos(4x-7y^2) + 2x\sin(4x-7y^2) + 2y\sin 2x}{14x^2y\cos(4x-7y^2) + \cos 2x}}$$

14. Find the derivative of y with respect to x. Show your work. (8 points)

$$y = (\tan 3x)^{2x}$$

$$\ln y = \ln (\tan 3x)^{2x}$$

$$\Rightarrow \ln y = 2x \ln (\tan 3x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 2x \cdot \frac{d}{dx} \ln(\tan 3x) + \ln(\tan 3x) \cdot \frac{d}{dx}(2x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 2x \frac{1}{\tan 3x} \cdot 3\sec^2 3x + 2\ln(\tan 3x)$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{6x\sec^2 3x}{\tan 3x} + 2\ln(\tan 3x)\right)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = (\tan 3x)^{2x}\left(\frac{6x\sec^2 3x}{\tan 3x} + 2\ln(\tan 3x)\right)}$$