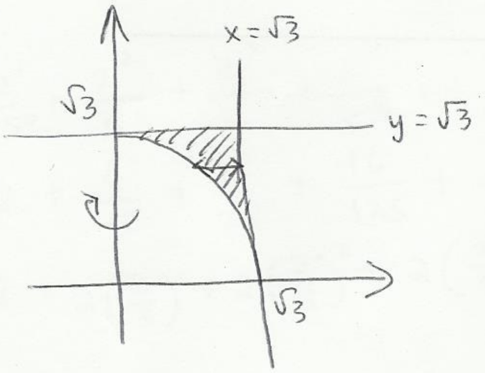
1. Find the volume of the solid generated by revolving the region in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$, and above by the line $y = \sqrt{3}$ about the y-axis. Show your work. (8 points)



$R(y) = \sqrt{3}$

$r(y) = \sqrt{3-y^2}$

$$\Rightarrow \quad V = \int_0^{\sqrt{3}} \pi\left(R(y)^2 - r(y)^2\right) dy$$

$$= \pi \int_0^{\sqrt{3}} \left(3 - (3-y^2)\right) dy$$

$$= \pi \int_0^{\sqrt{3}} y^2 \, dy$$

$$= \pi \left[\frac{y^3}{3}\right]_0^{\sqrt{3}} = \pi \, \frac{3\sqrt{3}}{3} = \boxed{\pi\sqrt{3}}$$

2. Evaluate the integral. Show your work. (8 points)

$$\int x \sec^2 x \, dx$$

$u = x$ $\quad du = dx$

$dv = \sec^2 x \, dx$ $\quad v = \tan x$

$$= x \tan x - \int \tan x \, dx$$

$$= \boxed{x \tan x - \ln|\sec x| + C}$$

3. Use trigonometric substitution to evaluate the integral. Show your work. (8 points)

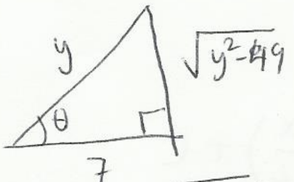
$$\int \frac{\sqrt{y^2 - 49}}{y} \, dy$$

$y = 7\sec\theta$

$dy = 7\sec\theta \tan\theta \, d\theta$

$y^2 - 49 = 49\sec^2\theta - 49$

$= 49\tan^2\theta$



$\tan\theta = \dfrac{\sqrt{y^2-49}}{7}$

$$= \int \frac{7\tan\theta}{\cancel{7}\cancel{\sec\theta}} \cancel{7}\cancel{\sec\theta} \tan\theta \, d\theta$$

$$= 7\int \tan^2\theta \, d\theta$$

$$= 7\int (\sec^2\theta - 1) \, d\theta$$

$$= 7\int \sec^2\theta \, d\theta - 7\int d\theta$$

$$= 7\tan\theta - 7\theta + C$$

$$= \cancel{7} \cdot \left(\frac{\sqrt{y^2-49}}{\cancel{7}}\right) - 7 \sec^{-1}\left(\frac{y}{7}\right) + C$$

$$= \boxed{\sqrt{y^2 - 49} - 7\sec^{-1}\left(\frac{y}{7}\right) + C}$$

4. Express the integrand as a sum of partial fractions and evaluate the integral. Show your work. (Hint: Factor x^4-16.) (8 points)

$$\int \frac{x^2}{x^4-16}\,dx$$

$$\frac{x^2}{x^4-16} = \frac{x^2}{(x^2-4)(x^2+4)} = \frac{x^2}{(x-2)(x+2)(x^2+4)}$$

let $$\frac{x^2}{(x-2)(x+2)(x^2+4)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$$

$$x^2 = A(x+2)(x^2+4) + B(x-2)(x^2+4) + (Cx+D)(x-2)(x+2)$$

$x=-2$

$$\cancel{4} = B(-4)(\cancel{8}) \Rightarrow B = -\frac{1}{8}$$

$x=2$

$$\cancel{4} = A(\cancel{4})(8) \Rightarrow A = \frac{1}{8}$$

$x=0$

$$0 = \frac{1}{\cancel{8}}\cdot\cancel{2}\cdot\cancel{4} - \frac{1}{\cancel{8}}(-\cancel{2})(\cancel{4}) + D(-2)(2)$$

$$0 = 1+1-4D \Rightarrow 4D = 2 \Rightarrow D = \frac{1}{2}$$

$x=1$

$$1 = \frac{1}{8}(3)(5) - \frac{1}{8}(-1)(5) + \left(C+\frac{1}{2}\right)(-1)(3)$$

$$1 = \frac{15}{8} + \frac{5}{8} - 3C - \frac{3}{2} \Rightarrow 3C = \frac{20}{8} - 1 - \frac{3}{2} = 1-1=0$$

$$C=0$$

$$\therefore \int \frac{x^2}{x^4-16}\,dx = \frac{1}{8}\int\frac{dx}{x-2} - \frac{1}{8}\int\frac{dx}{x+2} + \frac{1}{2}\int\frac{dx}{x^2+4}$$

$$= \frac{1}{8}\ln|x-2| - \frac{1}{8}\ln|x+2| + \frac{1}{2}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \boxed{\frac{1}{8}\ln\left|\frac{x-2}{x+2}\right| + \frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

5. Use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integral for convergence. Show your work. (6 points)

$$\int_1^\infty \frac{dx}{\sqrt{x^6+1}}$$

$x^6+1 > x^6 \quad , \quad x \in [1,\infty)$

$\sqrt{x^6+1} > \sqrt{x^6} = x^3$

$\frac{1}{\sqrt{x^6+1}} < \frac{1}{x^3}$

$\int_1^\infty \frac{1}{x^3}\,dx$ converges by the p-test.

$\therefore$ By direct comparison test

$\boxed{\int_1^\infty \frac{1}{\sqrt{x^6+1}}\,dx \text{ converges too}}$.

6. Find the limit of the sequence if it converges. Show your work. (6 points)

$$a_n = \frac{3^n}{n^3}$$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^n}{n^3} \quad \left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{3^n \ln 3}{3n^2} \quad \left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{3^n (\ln 3)^2}{6n} \quad \left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{3^n (\ln 3)^3}{6} \to \infty$

$\boxed{\text{It diverges.}}$

7. Write out the first few terms of the series to show how the series starts. Then find the sum of the series. Show your work. (6 points)

$$\sum_{n=0}^{\infty}\left(\frac{2^{n+1}}{5^n}\right)$$

$S = \frac{2^1}{5^0} + \frac{2^2}{5^1} + \frac{2^3}{5^2} + \frac{2^4}{5^3} + \ldots$

$S = 2 + \frac{4}{5} + \frac{8}{25} + \frac{16}{125} + \ldots$

$= 2 + 2\left(\frac{2}{5}\right) + 2\left(\frac{2}{5}\right)^2 + 2\left(\frac{2}{5}\right)^3 + \ldots$

$a = 2$

$r = \frac{2}{5} < 1$

$\therefore\ S = \frac{a}{1-r} = \frac{2}{1-\frac{2}{5}} = \frac{2}{\frac{3}{5}} = \boxed{\frac{10}{3}}$