1. Check for convergence or divergence of the series. Give reasons for your answer. Show your work. (8 points)

$$\sum_{n=1}^{\infty} \frac{4\tan^{-1} n}{1+n^2}$$

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Integral Test

$$\int_1^{\infty} \frac{4\tan^{-1}x}{1+x^2}\,dx$$

$$= \lim_{b\to\infty} \int_1^{b} \frac{4\tan^{-1}x}{1+x^2}\,dx \quad \text{Let } u=\tan^{-1}x$$

$$du = \frac{1}{1+x^2}\,dx$$

$$x=1 \Rightarrow u=\frac{\pi}{4}$$

$$x=b \Rightarrow u=\tan^{-1}b$$

$$= \lim_{b\to\infty} \int_{\pi/4}^{\tan^{-1}b} 4u\,du$$

$$= \lim_{b\to\infty} 2u^2 \Big|_{\pi/4}^{\tan^{-1}b}$$

$$= \lim_{b\to\infty} 2(\tan^{-1}b)^2 - 2\left(\tan\frac{\pi}{4}\right)^2$$

$$= 2\left(\frac{\pi}{2}\right)^2 - 2 = \boxed{\frac{\pi^2}{2} - 2}$$

$\therefore$ By integral Test $\sum_{n=1}^{\infty} \frac{4\tan^{-1}n}{1+n^2}$ converges.

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$$f(x) = \frac{4\tan^{-1}x}{1+x^2}$$

$$f'(x) = \frac{(1+x^2)\frac{4}{1+x^2} - (4\tan^{-1}x)2x}{(1+x^2)^2} = \frac{4 - 8x\tan^{-1}x}{(1+x^2)^2}$$

At $x=1$ $\quad \frac{4 - 8\cdot\frac{\pi}{4}}{4} = \frac{1-\frac{\pi}{2}}{4} < 0$

$\therefore$ $f(x)$ is a decreasing function.

2. Use any method to determine if the series converges or diverges. Give reasons for your answer. Show your work. (6 points)

$$\sum_{n=1}^{\infty} \frac{n \ln n}{2^n}$$

$$a_{n+1} = \frac{(n+1)\ln(n+1)}{2^{n+1}}$$

$$a_n = \frac{n \ln n}{2^n}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)\ln(n+1) \cdot 2^n}{2^{n+1} n \ln n} = \frac{\ln(n+1)}{2 \ln n} \cdot \frac{(n+1)}{n}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2} \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \cdot \lim_{n\to\infty} \frac{n+1}{n}.$$

$$= \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2} < 1$$

$\therefore \sum_{n=1}^{\infty} \frac{n \ln n}{2^n}$ converges.

3. Find the series' radius and interval of convergence. Check the convergence and divergence at the end points of the interval. Show your work. (8 points)

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(x-1)^{n+1}}{\sqrt{n+1}} \; \frac{\sqrt{n}}{(x-1)^n}\right|$$

$$= |x-1| \frac{\sqrt{n}}{\sqrt{n+1}}$$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} |x-1| \frac{\sqrt{n}}{\sqrt{n+1}} = |x-1|$$

For the power series to converge $|x-1| < 1$

$$\Rightarrow -1 < x-1 < 1$$

$$0 < x < 2$$

<u>At $x=0$</u> $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

By Alternating series test this converges.

<u>At $x=2$</u> $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. By p-series test with $p = 1/2$ this series diverges.

$\therefore$ Interval of convergence $=$ $[0,2)$

Radius of convergence $= 1$

4. Find the Taylor series generated by f at $x = a$. Show your work. (6 points)

$$f(x) = \frac{1}{(1-x)^2}, \quad a = 0$$

$f(x) = (1-x)^{-2}$

$f'(x) = 2(1-x)^{-3}$

$f''(x) = 2.3\,(1-x)^{-4}$

$f'''(x) = 2.3.4(1-x)^{-5}$

$\vdots$

$f^{(k)}(x) = 2.3.4.\ \cdots\ .(k+1)\,(1-x)^{-(k+2)}$

$f(0) = 1 = 1!$

$f'(0) = 2 = 2!$

$f''(0) = 2.3 = 3!$

$f'''(0) = 2.3.4 = 4!$

$\vdots$

$f^{(k)}(0) = 2.3.4 \cdots (k+1) = (k+1)!$

$\therefore \dfrac{1}{(1-x)^2} = f(0) + \dfrac{f'(0)}{1!}x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3 + \cdots + \dfrac{f^{(k)}(0)}{k!}x^k + \cdots$

$\Rightarrow \dfrac{1}{(1-x)^2} = 1 + \dfrac{2!}{1!}x + \dfrac{3!}{2!}x^2 + \dfrac{4!}{3!}x^3 + \cdots + \dfrac{(k+1)!}{k!}x^k + \cdots$

$= 1 + 2x + 3x^2 + 4x^3 + \cdots + (k+1)x^k + \cdots$

5. Find the first four terms of the binomial series for the given function. Also write an expression for the general n^{th} term. Show your work. (8 points)

$$(1+x^2)^{-\frac{1}{3}}$$

$$(1+x^2)^{-1/3} = 1 + \left(-\tfrac{1}{3}\right)x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}(x^2)^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}(x^2)^3 + \cdots$$

$$= 1 - \frac{x^2}{3} + \frac{4}{9.2!}x^4 - \frac{28}{27.3!}x^6 + \cdots$$

$$= 1 - \frac{x^2}{3} + \frac{2}{9}x^4 - \frac{14}{81}x^6 + \cdots$$

General $n^{\underline{th}}$ term

$$\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)\cdots\left(-\frac{1}{3}-(n-2)\right)}{(n-1)!}(x^2)^{n-1}$$

6. Assuming that the equations below define x and y implicitly as differentiable functions $x = f(t), y = g(t)$, find the slope of the curve $x = f(t), y = g(t)$ at the given value of t. Show your work. (Hint: Find dy/dt and dx/dt and remember $dy/dx = \frac{dy}{dt}\Big/\frac{dx}{dt}$). (6 points)

$$x \sin t + 2x = t, \quad t \sin t - 2t = y, \quad t = \pi$$

$x \sin t + 2x = t$

$\frac{d}{dt}(x \sin t) + \frac{d(2x)}{dt} = \frac{d(t)}{dt}$

$x \cos t + \sin t \frac{dx}{dt} + 2\frac{dx}{dt} = 1$

$\frac{dx}{dt}(2 + \sin t) = 1 - x\cos t$

$\frac{dx}{dt} = \frac{1 - x\cos t}{2 + \sin t}$

$\left.\frac{dx}{dt}\right|_{t=\pi} = \frac{1+x}{2}$

$t \sin t - 2t = y.$

$\frac{d}{dt}(t \sin t) - \frac{d}{dt}(2t) = \frac{dy}{dt}$

$t\cos t + \sin t - 2 = \frac{dy}{dt}$

$\left.\frac{dy}{dt}\right|_{t=\pi} = -\pi - 2$

$\therefore \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \boxed{\cancel{\frac{1+x}{2(-\pi-2)}}}$ $\quad \frac{-\pi-2}{\frac{1+x}{2}} = \frac{-2\pi-4}{1+x}$

At $t = \pi$

$x \sin\pi + 2x = \pi \Rightarrow 2x = \pi$

$x = \frac{\pi}{2}$

At $t = \pi$

$\pi \sin\pi - 2\pi = y \Rightarrow y = -2\pi$

$\therefore \frac{dy}{dx}$ at $t = \pi = \cancel{\frac{1+\frac{\pi}{2}}{2(-\pi-2)}} = \cancel{\frac{2+\pi}{-4(\pi+2)}} = \boxed{\cancel{-\frac{1}{4}}}$ $\quad \frac{-2\pi-4}{1+\frac{\pi}{2}} = \frac{-4\pi-8}{2+\pi} = \boxed{-4}$

7. Find the equivalent Cartesian equation for the given polar equation. Then describe or identify the graph. Show your work. (8 points)

$$r = 2\cos\theta + 2\sin\theta$$

$r = \sqrt{x^2+y^2}$ $\quad r^2 = x^2+y^2$ $\quad x = r\cos\theta$ $\quad y = r\sin\theta$

$\frac{x}{r} = \cos\theta$ $\quad \frac{y}{r} = \sin\theta$

$$r = \frac{2x}{r} + \frac{2y}{r}$$

$$\Rightarrow r^2 = 2x + 2y$$

$$\Rightarrow x^2 + y^2 = 2x + 2y.$$

$$\Rightarrow (x^2 - 2x) + (y^2 - 2y) = 0$$

$$\Rightarrow (x^2 - 2x + 1) + (y^2 - 2y + 1) = 2$$

$$\Rightarrow \boxed{(x-1)^2 + (y-1)^2 = 2}$$

$\therefore$ It is circle at $(1,1)$ with radius $\sqrt{2}$.